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A NUMERICAL METHOD FOR SOLVING PARABOLIC EQUATIONS BASED ON THE USE OF A MULTIGRID TECHNIQUES

OLGA MILYUKOVA, MARINA LADONKINA, AND VLADIMIR TISHKIN

Abstract. A numerical method for solving parabolic equations based on multigrid techniques is proposed. The stability, approximation and conservation properties of the method are investigated theoretically and numerically for several initial-boundary model problems for the heat conduction equation. The use of the method makes it possible to considerably reduce the computational work as compared to either implicit or explicit schemes. A parallel implementation of the method is presented.

Key words. parabolic equations, multigrid method, stability and accuracy, conservation property, parallel computations

1. Introduction

In the numerical modeling of many physical problems, for instance, in fluid dynamics, the diffusion processes should be taken into account. Therefore, the need in solving parabolic equations arises. The use of explicit schemes for the approximation of parabolic type equations implies severe limitations on the time stepsize due to stability conditions [1] which results in very time-consuming computations. The implicit schemes can be used with larger time stepsizes, but the additional computational costs due to the solution of arising linear algebraic systems may also result in spending a lot of computational time.

For the solution of the linear algebraic systems arising in the implicit schemes, the multigrid method can be used. The latter was first proposed by R.P.Fedorenko for the solution of elliptic equiations in [2], and then justified theoretically by N.S.Bakhvalov in [3]. The generalization of the multigrid method for the case of parabolic equations was presented in [4], [5]. However, the use of the multigrid methods in their standard form may also involve a big amount of computational work, which may result in a marginal, if any, improvement as compared to the calculations using explicit schemes. For instance, such situation arises in the numerical investigation of Richtmyer-Meshkov instability [6] under experimental conditions. This explains the need in development of new numerical methods for the solution of parabolic type equations, especially for the cases when detailed spatial grids are used.

In papers [7], [8], an efficient algorithm for the solution of initial-boundary problems for parabolic equations is proposed which is based on the use of a two-grid method. The processing of each time layer involves only one iteration of the two-grid cycle and only few (or even one) smoothing iterations for each grid level. A theoretical and numerical investigation of several model initial-boundary problems for the heat conduction equation showed that the proposed algorithm possesses the same accuracy and stability as the fully implicit scheme on the finest grid.

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Moreover, in contrast with the computational scheme presented in [7], the scheme developed in [8] possesses the conservation property. In [9], the new numerical method for the solution of two-dimensional parabolic equations is proposed, based on the use of L-grid method. The choice of the coarsest grid is defined by the conditions $h^2_{x,L} = o(\tau)$, $h^2_{y,L} = o(\tau)$, where $h_{x,L}, h_{y,L}$ are the spatial stepsizes for the coarsest grid, and τ is the time stepsize. At each time layer, only one iteration of the L-grid cycle is performed, and only few (or even one) smoothing iterations are done for each grid level. For a two-dimensional model problem for the heat conduction equation, it is shown that the proposed method allows to obtain a smooth solution (having bounded finite-difference derivatives up to the fourth order) with the same order of accuracy as that of the fully implicit scheme. This method is suitable for the solution of problems which require the time stepsize essentially larger than $O(h^2)$, where $h = \max(h_{x,1}, h_{y,1})$ is the maximum stepsize of the finest spatial grid. A parallel implementation of the proposed method is developed.

In the present paper, the numerical method based on the use of L grids for the solution of two-dimensional parabolic equations [9] is considered. This method is generalized for the three-dimensional case. For a two-dimensional model problem for the heat conduction equation, it is shown that under some conditions of sufficient smoothness of the initial data and the stability condition, the new numerical method allows to obtain the solution with the same order of accuracy as that of the fully implicit scheme. It is demonstrated that the use of the proposed method allows to reduce substantially the arithmetic work and the run-time as compared to the use of the implicit or the explicit schemes on the finest grid. Numerical tests for model problems with smooth coefficients has confirmed the good accuracy of the method. Numerical results obtained for a model problem with discontinuous coefficient has confirmed that this method finds the solution with the relative error of several per cent related to the solution obtained with the use of the fully implicit scheme.

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2. Algorithm of new numerical method for the solution of parabolic equations

The design and analysis of the algorithm implementing the new numerical method for the solution of parabolic equations is demonstrated for the case of initial-boundary problem for the heat equation

$$
\begin{aligned}
&\rho C_v \frac{\partial u}{\partial t} = \operatorname{div}(\kappa \operatorname{grad} u) + f, \quad (\overrightarrow{x}, t) \in G, \\
&u(\overrightarrow{x}, t) = g(\overrightarrow{x}, t) \quad \text{if} \quad \overrightarrow{x} \in \gamma, u(\overrightarrow{x}, 0) = T_0(\overrightarrow{x}),
\end{aligned}
\tag{1}
$$

where C_v is the heat capacity coefficient at constant volume, ρ is the density, κ is the heat conduction coefficient, u is the temperature at the point $\overrightarrow{x} = (x, y)$ or $\overrightarrow{x} = (x, y, z)$ at the time t, $G = \{0 < x < l_1, 0 < y < l_2, 0 < t \leq T\}$ or $G = \{0 < x < l_1, 0 < y < l_2, 0 < z < l_3, 0 < t \leq T\}$, γ is the boundary of the computational domain, f is the heat source density, and $g(\overrightarrow{x}, t)$ and $T_0(\overrightarrow{x})$ are given functions. Here κ is a positive piecewise continuous scalar function. In order to construct the finite-difference approximation of the problem (1), we will use the fully implicit scheme on the uniform grid with the stepsizes h_x, h_y, h_z along the x, y, z directions and the time stepsize τ. The discrete problem can be written as the sequence of systems of linear algebraic equations $A_h u^{n+1} = f_h$ with the unknown