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NUMERICAL STUDY OF TIME-PERIODIC SOLITONS IN THE DAMPED-DRIVEN NLS

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Abstract. We study localised attractors of the parametrically driven, damped nonlinear Schrödinger equation. Time-periodic solitons of this equation are obtained as solutions of the boundary-value problem on a two-dimensional domain. Stability and bifurcations of periodic solitons and their complexes is classified. We show that the bifurcation diagram can be reproduced using a few-mode approximation.

Key words. Nonlinear Schrödinger equation, temporally periodic solitons, newtonian iterative scheme, numerical continuation, stability, bifurcations

1. Introduction

We investigated the parametrically driven damped nonlinear Schrödinger equation (NLS),

$$i\psi_t + \psi_{xx} + 2|\psi|^2\psi - \psi = h\psi^* - i\gamma\psi. \tag{1}$$

that describes a large number of resonant phenomena in various physical media: nonlinear Faraday resonance in a vertically oscillating water trough [15],[16],[23]; the effect of phase-sensitive amplifiers on solitons in optical fibers [13],[18],[14]; magnetization waves in an easy-plane ferromagnet placed in a combination of a static and microwave field [4]; the amplitude of synchronized oscillations in vertically vibrated pendula lattices [12],[2],[11] etc. More applications of Eq.(1) are listed in [8, 9].

In Eq. (1), $\gamma > 0$ is the damping coefficient, $h > 0$ the amplitude of the parametric driver, and symbol "*" means the complex conjugation.

Equation (1) exhibits different classes of soliton solutions existing on the (h, γ)-plane above the straight line $h = \gamma$.

Two stationary solitons ψ_+ and ψ_- are available in analytic form [4]:

$$\psi_\pm(x) = A_\pm e^{-i\theta_\pm}\operatorname{sech}(A_\pm x), \tag{2}$$

where

$$A_\pm = \sqrt{1 \pm \sqrt{h^2 - \gamma^2}},$$
$$\theta_+ = \frac{1}{2}\arcsin\frac{\gamma}{h}, \qquad \theta_- = \frac{\pi}{2} - \theta_+.$$

The soliton $\psi_-(x)$ is known to be unstable for all h and γ. Stability properties of the soliton $\psi_+(x)$ for various h and γ were examined in [4].

Other localised attractors of Eq. (1) (that have been found in numerical simulations) include: stationary multi-soliton complexes [5], uniformly travelling solitons and complexes [6, 7], time-periodic and quasi-periodic solitons [1, 10].

In this paper, we study time-periodic attractors of Eq. (1) that arise as a Hopf bifurcation of stable stationary soliton solutions. Attractors of periodic solitons on the (h, γ)-plane were obtained in [10] on the basis of direct numerical simulation

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of Eq. (1) with initial condition in the form of stationary soliton ψ_+. In [22, 20] these were reobtained as solutions of a two-diemsnional boundary-value problem for Eq. (1). Here, we employ the same numerical approach for our numerical analysis of time-periodic solitons. Our purpose is to clarify transformations and interconnections between coexisting periodic one- and two-soliton branches in the region of parameter $\gamma \gtrsim 0.35$.

In Section 2, we formulate the 2D boundary-value problem and describe our numerical approach. Results of numerical study are discussed in Section 3. We present the branches of time-periodic one- and two-soliton colutions for $\gamma = 0.35$. Also, we demonstrate the spatially nonsymmetric time-periodic two-soliton complex for $\gamma = 0.41$. In Section 4, a simple few-mode approximation of the 2D nonlinear boundary value problem has been suggested. Main results have been summarized in Section 5.

2. Numerical approach

2.1. Formulation of 2D boundary-value problem. We consider the time-periodic solutions Eq. (1) as solutions of the boundary value problem on the two-dimensional domain $(-\infty, \infty) \times (0, T)$. The boundary conditions are

$$\psi(x,t) = 0 \quad \text{as } x \to \pm\infty, \quad \text{and} \quad \psi(x, t+T) = \psi(x,t). \tag{3}$$

The 2D boundary-value problem (1),(3) is solved numerically for the unknown time-periodic function $\psi(x,t)$, where the period T is also unknown.

Letting $\tilde{t} = t/T$ $(0 < \tilde{t} < 1)$ and defining $\tilde{\psi}(x,\tilde{t}) = \psi(x,t)$, the boundary-value problem (1),(3) can be reformulated on the rectangle $(-L, L) \times (0, 1)$ (where L is chosen to be sufficiently large):

$$\begin{cases} \mathbf{F} \equiv i\tilde{\psi}_{\tilde{t}}(x,\tilde{t}) + T\Phi(\tilde{\psi}(x,\tilde{t}), h, \gamma) = 0, \\ \tilde{\psi}(\pm L, \tilde{t}) = 0, \\ \tilde{\psi}(x,0) = \tilde{\psi}(x,1). \end{cases} \tag{4}$$

Here,

$$\Phi(\tilde{\psi}(x,\tilde{t}), h, \gamma) = \tilde{\psi}_{xx} + 2|\tilde{\psi}|^2\tilde{\psi} - \tilde{\psi} - h\tilde{\psi}^* + i\gamma\tilde{\psi}. \tag{5}$$

Eq. (4) is supplemented with an additional equation borrowed from [19]:

$$\mathrm{Re}\Phi(\tilde{\psi}(x^*, \tilde{t}^*), h, \gamma) = 0, \quad x^* = t^* = 0. \tag{6}$$

Solutions $(T, \tilde{\psi})$ of the 2D boundary-value problem (4-6) were path-followed in h for the fixed γ. The time-independent solution at the point of Hopf bifurcation is used as starting point of the continuation process. At each value of parameter h we employ Newtonian iteration scheme presented in 2.2. Continuation algorithm is described in 2.3.

In what follows, we omitted tildes above ψ and t.

For the graphical representation of solutions we are using the averaged energy defined by

$$\bar{E} = \frac{1}{T}\int_0^T dt \int_{-\infty}^{\infty} dx\, E(x,t), \tag{7}$$

where

$$E(x,t) = |\psi_x|^2 + |\psi|^2 - |\psi|^4 + h\mathrm{Re}(\psi^2). \tag{8}$$

Note that the energy $\int E dx$ is *not* an integral of motion for $\gamma \neq 0$.

Stability of solutions is classified by examining the Floquet multipliers of the corresponding linearized equation. Details are in [22, 8].