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A NEW INCOMPRESSIBLE SMOOTHED PARTICLE HYDRODYNAMICS-IMMERSED BOUNDARY METHOD

RODRIGUE KÉOU NOUTCHEUWA AND ROBERT G. OWENS

Abstract. In this article we develop a new smoothed particle hydrodynamics (SPH) method suitable for solving the incompressible Navier-Stokes equations, even with singular forces. Singular source terms are handled in a manner similar to that in the immersed boundary (IB) method of Peskin (2002). The numerical scheme implements a second-order pressure-free projection method due to Kim and Moin (1985) and completely obviates the difficulties that may be faced in prescribing Neumann pressure boundary conditions. The proposed SPH method is first tested on the planar start-up Poiseuille problem and a detailed error analysis is performed. For this problem, the results are similar whether the SPH particles are free to move or fixed on a regular grid. Our hybrid SPH-IB method is then used to calculate the dynamics of a stretched immersed elastic membrane and the advantages in this case of fixing the SPH particles, rather than allowing them to move with the fluid, are discussed.

Key words. Smoothed particle hydrodynamics, incompressible flow, projection method, singular force, immersed boundary

1. Introduction

The SPH method permits the numerical solution of the equations of fluid dynamics or solid mechanics by representing the fluid/solid as an ensemble of fluid/solid particles. The method is mesh-free and based on a Lagrangian formulation. The method was first described in 1977 by Gingold and Monaghan [19] and independently in the same year by Lucy [37]. The motivation behind the creation of this method was the need to solve complex problems that traditional mesh-based Eulerian methods (such as finite difference methods, finite element methods or finite volume methods) were not able to handle easily. Problems may be encountered when these mesh-based methods are used for the solution of problems involving free surfaces, deformable boundaries and problems involving pronounced deformations. Moreover, when the computational domain is complex it is not always clear how to construct meshes for mesh-based methods that take into account all the irregularities of the domain, even if techniques such as domain decomposition are used. Originally, the SPH method was introduced with the aim of simulating astrophysical phenomena that introduce important variations in the density in complex and non-symmetric geometries [40]. Since its first appearance in the literature the method has been further developed and applied to the solution of many different problems, among which we could cite multi-phase flows [10, 24, 43], free-surface flows [2, 55], impacts and explosions [35, 44], heat conduction modelling [8], viscoelastic flows [16, 17, 47] and solid mechanics [14, 38, 65]. Many other application areas are described in the recent extensive reviews of Cleary et al. [9] and Monaghan [42]. However, there are a great many examples in the scientific literature of SPH methods being tested on problems where, traditionally, mesh-based methods have been successful. For example, SPH methods have been used to solve for flow around right circular cylinders and other bluff bodies [15, 23, 29, 57], steady and

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start-up Poiseuille flow [4, 23, 47, 57] and lid-driven cavity flow [6, 29, 64]. The present paper and a recent article using the SPH method for the simulation of polymeric fluid flows [47] are the only examples in the literature that we know of that use the SPH method to solve the incompressible Navier-Stokes equations with a distribution of singular forces.

When simulating incompressible fluid flows a significant challenge consists of ensuring that the incompressibility condition is satisfied. In order to accomplish this the traditional SPH method, often termed the Weakly Compressible Smoothed Particle Hydrodynamics (WCSPH) method (see, for example, [1, 13, 19, 39, 40, 42, 44, 64]), supposes that the fluid is weakly compressible and the density of the fluid (more particularly, of each fluid particle) is calculated at each time step. The most common technique adopted for approximating the incompressibility of the fluid is to use an equation of state that relates the pressure and the density. Although this equation of state may have several different forms (see the articles cited above) the key element is that the magnitude of the speed of sound used in the equation must be sufficiently small to be practical whilst at the same time being sufficiently large in order to maintain the density approximately constant from one time step to another [4]. The use of a large magnitude speed of sound implies a very severe CFL condition on the time step in the calculations [13, 17, 23, 29, 45, 64]. This approximate enforcement of the incompressibility condition introduces errors that have as their source small fluctuations in the density [13, 55] and may give rise to large pressure oscillations [29, 64].

An alternative approach leading to a truly incompressible SPH method involves the use of a projection (operator splitting) scheme, originally developed in the context of Eulerian mesh-based methods, in order to impose the incompressibility condition in the calculations. The original (first-order) projection scheme was introduced by Chorin [7] in 1968 and has been extensively used since then. The projection method was used for the first time in the context of SPH methods by Cummins and Rudman [13] who gave their scheme the name Projection Smoothed Particle Hydrodynamics (PSPH): an acronym that we will incorporate into the names of our own schemes. In the SPH calculations of Lee et al. [29] involving flow around a square cylinder, lid-driven cavity flow and a dam-breaking problem, their truly incompressible SPH method was found in all cases to yield smoother velocity and pressure profiles than was possible with the WCSPH method. Moreover, the CPU time was anywhere between 2 and 20 times less with the incompressible SPH method than was possible with the WCSPH approach. An important problem with the Neumann pressure boundary condition used in many pressure-correction projection schemes and present, for example, in the work of Cummins and Rudman [13] and Lee et al. [29], has been highlighted by Brown et al. [5], Guermond et al. [21], Hosseini and Feng [23] and others. Typically, zero Neumann boundary conditions are imposed for the Poisson equation satisfied by the pressure. However, in open-boundary problems or when flow is around bluff bodies or through a channel of variable cross section this is not correct and leads to numerical boundary layers. Hosseini and Feng [23] avoided this problem by using the rotational projection scheme of Timmermans et al. [60] and imposing a non-homogeneous Neumann condition on the pressure which was shown to be consistent with the linear momentum equation.

The problems plaguing the accuracy and stability of SPH computations due to particle clustering (anisotropic particle spacing) have been well documented in the scientific literature (see, for example, [18, 25, 64]). Monaghan [41] demonstrated