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A SPARSE APPROXIMATE INVERSE PRECONDITIONER FOR NONSYMMETRIC LINEAR SYSTEMS

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Abstract. Motivated by the paper [16], where the authors proposed a method to solve a symmetric positive definite (SPD) system $Ax = b$ via a sparse-sparse iterative-based projection method, we extend this method to nonsymmetric linear systems and propose a modified method to construct a sparse approximate inverse preconditioner by using the Frobenius norm minimization technique in this paper. Numerical experiments indicate that this new preconditioner appears more robust and takes less time of constructing than the popular parallel sparse approximate inverse preconditioner (PSM) proposed in [6]

Key words. nonsymmetric linear systems, preconditioning, projection method, Krylov subspace methods, PSM, sparse approximate inverse

1. Introduction

Consider the solution of a sparse nonsymmetric linear system of algebraic equations:

$$Ax = b, \tag{1}$$

where $A \in R^{n\times n}$ is a nonsingular matrix, $x \in R^n$ is an unknown vector, and $b \in R^n$ is a given vector. This system arises in many areas of scientific computing, such as in fluid mechanics [9], solid mechanics [4], and fluid flow in porous media [5]. When A is large and sparse, direct solvers such as Gauss elimination may bring 'fill-in' phenomenon and require huge amount of work and memory storage. Another approach to solve this system uses Krylov subspace iterative methods such as the generalized minimal residual method (GMRES) and the biconjugate gradient stabilized method (BiCGSTAB) [15, 18]. These methods require less storage but the rate of convergence depends strongly on the spectral distribution of matrix A. Usually, the more clustered the eigenvalues of A are, the faster these methods converge. Toward this end, we may apply a preconditioning technique; i.e., we may transform system (1) into the following system:

$$AMy = b, \quad x = My \quad \text{or} \quad MAx = Mb, \tag{2}$$

where M is a nonsingular matrix and is required to be cheaply constructed, called a preconditioner. If $M \approx A^{-1}$, the coefficient matrix AM of system (2) always has a 'good' spectral distribution, and then using the Krylov subspace iterative methods for solving (2), we can achieve much faster convergence.

Recently, many preconditioning techniques have been developed; see, e.g., [2, 15] for a review. In this paper, we focus on a sparse approximate inverse technique based on minimizing the Frobenius norm [6, 7, 11, 12, 13, 14]. Because of the inherent parallel feature of this technique, it has attracted much attention. Its basic idea is to construct a sparse nonsingular matrix by the constrained minimization

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problem:

$$\min_{M \in \wp} \|AM - I\|_F, \tag{3}$$

where $\wp$ is a set of sparsity pattern of matrices, $\|.\|_F$ denotes the Frobenius norm of a matrix, and I denotes the identity matrix. The minimization problem (3) can be decoupled into n independent linear least squares problems:

$$\min_{M \in \wp} \|AM - I\|_F = \min_{M \in \wp} \sum_{j=1}^{n} \|Am_j - e_j\|_2, \tag{4}$$

where m_j and e_j denote the jth column of M and I, respectively. Thus we can construct the preconditioner M by solving n independent linear least squares problems. However, how to choose a 'good' sparsity pattern of M that can be effectively constructed is still challenging. The aim of this paper is to construct a desired preconditioner M for the sparse nonsymmetric system (2), and we will discuss the construction process in the following sections in detail. Numerical experiments indicate that this new preconditioner appears more robust and takes less time of constructing than the popular parallel sparse approximate inverse preconditioner proposed in [6].

The paper is organized as follows. In Section 2, we briefly describe three basic algorithms for computing approximate solutions of nonsymmetric systems. Then, in Section 3, we develop a new method to construct the preconditioner we are proposing. Finally, numerical experiments to check this preconditioner's effectiveness are presented in Section 4.

2. Approximate Solutions of Nonsymmetric Systems

In this section, we extend the method which was used in [16] for solving SPD linear systems to general nonsymmetric systems and then modify this method so that it can be more flexible and effective.

Let $A \in R^{n\times n}$ be a general nonsingular matrix. Also, let K and L be two m-dimensional subspaces of R^n, and $x_0 \in R^n$ be an initial guess of the solution of system (1). A projection method is a process which finds an approximate solution $x \in R^n$ of (1) as follows:

$$\text{Find } x \in x_0 + K \;\text{ such that }\; b - Ax \perp L. \tag{5}$$

Now, let $K = \text{span}\{e_{i_1}, e_{i_2}, ..., e_{i_m}\}$ and $L = AK$, where e_{i_j} is the i_jth column of the identity matrix and m is a small integer. Then problem (5) can be transformed into the following form:

$$\begin{aligned} &\text{Find } x \in x_0 + Ey \;\text{ such that }\; r_0 - AEy \perp L, \\ &\qquad\qquad\qquad \text{i.e., } (E^T A^T AE)y = E^T A^T r_0, \end{aligned} \tag{6}$$

where $E = [e_{i_1}, e_{i_2}, ..., e_{i_m}]$, $r_0 = b - Ax_0$, and $y \in R^m$.

If we loop (6) and use it to solve the linear systems:

$$Am_j = e_j, \tag{7}$$

then we define a new approach for solving the systems in (7) as follows:

Algorithm 1 (sparse approximate solution to the system $Am_j = e_j$):

1. Choose an initial guess m_j and compute $r = e_j - Am_j$;
2. For $i = 1 : n_p$,