INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING, *SERIES B*
Volume 5, Number 4, Pages 374–399

MULTISCALE FEM-FVM HYBRID METHOD FOR CONVECTION-DIFFUSION EQUATIONS WITH PERIODIC DISCONTINUOUS COEFFICIENTS IN GENERAL CONVEX DOMAINS

LIRUI SHEN, LIQUN CAO, AND YAU SHU WONG

Abstract. This paper presents the multiscale analysis and numerical algorithms for the convection-diffusion equations with rapidly oscillating periodic discontinuous coefficients. The multiscale asymptotic expansions are developed and an explicit rate of convergence is derived for the convex domains. An efficient multiscale hybrid FEM-FVM algorithm is constructed, and numerical experiments are reported to validate the predicted convergence results.

Key words. Convection-diffusion equation, convex domain, periodic discontinuous coefficients, composite materials, porous media, multiscale asymptotic expansion, finite element method, finite volume element method.

1. Introduction

In this paper, we consider the convection-diffusion equations with rapidly oscillating periodic discontinuous coefficients given as follows:

$$(1)\quad \begin{cases} \mathcal{L}_\varepsilon(u^\varepsilon) \equiv \dfrac{\partial u^\varepsilon(x,t)}{\partial t} - \dfrac{\partial}{\partial x_i}\Big(a_{ij}^\varepsilon(x,t)\dfrac{\partial u^\varepsilon(x,t)}{\partial x_j}\Big) + \dfrac{\partial}{\partial x_i}\Big(b_i^\varepsilon(x,t)u^\varepsilon(x,t)\Big) + \\ \qquad\qquad a_0^\varepsilon(x,t)u^\varepsilon(x,t) = f(x,t), \quad (x,t) \in \Omega\times(0,T), \\ u^\varepsilon(x,t) = g_0(x,t), \qquad (x,t)\in\Gamma_0\times(0,T), \\ \sigma_\varepsilon(u^\varepsilon) \equiv \nu_i\Big(a_{ij}^\varepsilon(x,t)\dfrac{\partial u^\varepsilon(x,t)}{\partial x_j} - b_i^\varepsilon(x,t)u^\varepsilon(x,t)\Big) = g_1(x,t), \\ \qquad\qquad (x,t)\in\Gamma_1\times(0,T), \\ u^\varepsilon(x,0) = \bar{u}_0(x), \qquad x\in\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ is a bounded convex Lipschitz domain with the boundary $\partial\Omega = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, $\text{meas}(\Gamma_0) > 0$, where $\text{meas}(\Gamma_0)$ denotes the Lebesgue's measure of Γ_0; $u^\varepsilon(x,t)$ is the unknown function, $f(x,t)$, $g_0(x,t)$, $g_1(x,t)$ and $\bar{u}_0(x)$ are known functions. Here, we focus on the following specified cases: $a_{ij}^\varepsilon(x,t) = a_{ij}(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^k})$, $b_i^\varepsilon(x,t) = b_i(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^k})$ and $a_0^\varepsilon(x,t) = a_0(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^k})$, $k = 0, 1$; $\nu = (\nu_1, \cdots, \nu_d)$ is the outward unit normal to Γ_1. Throughout the paper, the Einstein summation convention on repeated indices is adopted. By C we denote a positive constant independent of ε.

Let $\xi = \varepsilon^{-1}x$, $\tau = \varepsilon^{-k}t$, $k = 0, 1$. We make the following assumptions:

$(\mathbf{A_1})$ For $k = 0$, $a_{ij}(\xi, t)$, $b_i(\xi, t)$ and $a_0(\xi, t)$ are 1-periodic in ξ; For $k = 1$, $a_{ij}(\xi, \tau)$ $b_i(\xi, \tau)$ and $a_0(\xi, \tau)$ are 1-periodic in ξ and τ_0-periodic in τ.

Received by the editors May 22, 2014.

2000 *Mathematics Subject Classification.* 65F10, 65W05.

This work is supported by the National Natural Science Foundation of China (grant # 91330202), National Basic Research Program of China (973 Program) (grant # 2010CB832702), Natural Sciences and Engineering Research Council of Canada and project supported by the Funds for Creative Research Groups of China (grant # 11321061).

($\mathbf{A_2}$) a_{ij}, b_i, $a_0 \in L^\infty(\mathbb{R}^d \times (0,+\infty))$; $a_{ij} = a_{ji}$; there are two positive constants γ_0 and γ_1 such that $\gamma_0\eta_i\eta_i \leq a_{ij}(\xi,\tau)\eta_i\eta_j \leq \gamma_1\eta_i\eta_i$, $\forall\, \eta = (\eta_1, \cdots, \eta_d) \in \mathbb{R}^d$ for *a.e.* $\xi \in \mathbb{R}^d$, $\tau \in (0,+\infty)$.

($\mathbf{A_3}$) $f \in L^2(0,T;L^2(\Omega))$, $g_0 \in L^2(0,T;H^{1/2}(\Gamma_0))$, $g_1 \in L^2(0,T;H^{-1/2}(\Gamma_1))$ and $\bar{u}_0 \in H^1(\Omega)$.

($\mathbf{A_4}$) Let $Q = (0,1)^d$ be the reference cell of the composite material. Assume that $Q \subset\subset Q'$ with $Q' = \bigcup_{m=1}^{L} \overline{D}_m$. Suppose that the boundaries of D_m belong to $C^{1,\gamma}$ for some $0 < \gamma < 1$ and that $a_{ij}(\xi,\tau) \in C^\mu(\overline{D}_m \times (0,+\infty))$, $b_i(\xi,\tau) \in C^\mu(\overline{D}_m \times (0,+\infty))$, $i,j = 1, \cdots, d$, for some constants $0 < \mu < 1$.

Problem (1) has a wide range of applications in fluid mechanics especially in highly heterogeneous media; and in the heat and mass transfer problems for composite materials or porous media (see [4, 22, 34]). It is noted that the problem often involves materials with a large number of heterogeneities (inclusions or holes). For homogenization results concerning linear parabolic equations with rapidly oscillating coefficients which depend on the spatial and time variables, we refer to Bensoussan, Lions and Papanicolaou [4] for the periodic cases and to Colombini and Spagnolo [17] for the general non-periodic case. For a type of nonlinear parabolic partial differential operators, Pankov [33] and Svanstedt [36] derived the G-convergence and the homogenization results. Zhikov, Kozlov and Oleinik [41] investigated the parabolic operators with almost periodic coefficients and presented convergence results for the asymptotic homogenization.

By introducing a cutoff function, Bensoussan, Lions and Papanicolaou [4] obtained the strong convergence result without an explicit rate for the first-order corrector of the linear parabolic equations in $L^2(0,T;H^1(\Omega))$. Brahim-Otsmane, Francfort and Murat [5] extended this result to $L^2(0,T;W^{1,1}(\Omega))$. Ming and Zhang [30] derived the convergence result with an explicit rate $\varepsilon^{1/2}$ for the case $k = 0$ under the assumption $u^0 \in H^{3,1}(\Omega \times (0,T))$, where $u^0(x,t)$ is the solution of the linear homogenized parabolic equation. Allegretto, Cao and Lin [2] investigated the higher-order multiscale method for the linear parabolic equations in the four specific cases $k = 0,1,2,3$, and obtained the convergence results with an explicit rate $\varepsilon^{1/2}$ under the assumption $u^0 \in H^{s+2,1}(\Omega \times (0,T))$, $s = 1,2$. It is well known that, for a bounded convex polygonal Lipschitz domain Ω, the assumptions $u^0 \in H^{s+2,1}(\Omega \times (0,T))$, $s = 1,2$ may be invalid. Thus the error estimates presented in [2] are not valid. On the other hand, many results are now available for the finite volume element methods (FVEM) for the elliptic and parabolic equations. For example, we refer to [3, 7, 27, 11, 14, 15, 18, 19, 20] for the early important results. Chou and Ye [13] presented the unified variational form of the conforming and nonconforming elements, and the DG for 2-D elliptic equations with the error estimates of triangular elements. Li et al. [25] and Luo et al. [26] developed FVEM and the stabilized FVEM for the Navier-Stokes equations in 2-D, and obtained the error estimates for triangular elements under the assumption $\Delta t = O(h)$. However, the error estimates for 3-D are not available. Under the assumption that the term of right side $f(x,t) \equiv 0$ is indispensable, Sinha and Geiser [35] investigated the FVEM for the 2-D and 3-D convection-diffusion equations, and derived the optimal error estimates for the triangular and tetrahedral elements. For other recent results in this topic, we refer to [31, 21, 29, 40].

In this paper, we present the following new results:

(i) The interior error estimates for the multiscale asymptotic solutions of the original problem (1) are obtained under the weaker assumptions (see Theorem 3.1).