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CHARACTERIZATION AND CONSTRUCTION OF LINEAR SYMPLECTIC RK-METHODS*1)

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Abstract

A characterization of linear symplectic Runge-Kutta methods, which is based on the W-transformation of Hairer and Wanner, is presented. Using this characterization three classes of high order linear symplectic Runge-Kutta methods are constructed. They include and extend known classes of high order linear symplectic Runge-Kutta methods.

1. Introduction

The present paper is a continuation of [13] where characterizations of symmetric and symplectic Runge-Kutta methods, based on the W-transformation of Hairer and Wanner, were presented. Using the characterization of symplectic Runge-Kutta methods, two classes of high order symplectic Runge-Kutta methods were constructed there. In the present paper we shall discuss a characterization of linear symplectic Runge-Kutta methods, which is based on the W-transformation of Hairer and Wanner. Up to now only symmetric one-step methods are found to be linear symplectic in the class of high order one-step methods. We shall construct three classes of high order linear symplectic Runge-Kutta methods, which include and extend known classes of high order linear symplectic Runge-Kutta methods. In this paper we shall continue to use the notation in [13].

It is well known that the stability function of implicit Runge-Kutta methods may be written as

$$R(z)=\frac{\det(I-zA+zeb^T)}{\det(I-zA)}, \tag{1.1}$$

or

$$R(z)=1+zb^T(I-zA)^{-1}e. \tag{1.1'}$$

In [6] Feng has proved that the necessary and sufficient condition of linear symplectic schemes is

$$R(z)R(-z)=1. \tag{1.2}$$

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From [6] we can easily obtain that symmetric Runge-Kutta methods are linear symplectic. In [13] we have proved

Theorem 1.1. *An s-stage RK-method with distinct nodes c_i and $b_i \neq 0$ satisfying $B(p), C(\eta)$ and $D(\zeta)$ with $p \geq s+\zeta$ is symmetric if and only if*

a) *$\widetilde{P}c = e - c$ for the permutation matrix $\widetilde{P}$,*

b) *the transformation matrix X of the method takes the following form*

$$X = W^TBAW = \begin{pmatrix} 1/2 & -\xi_1 & & & \\ \xi_1 & \ddots & \ddots & & \\ & \ddots & 0 & -\xi_\nu & \\ & & \xi_\nu & \multicolumn{2}{|c}{R_\nu} \end{pmatrix}, \quad \text{where } \nu = \min(\eta, \zeta) \tag{1.3}$$

having the residue matrix R_ν whose (k,l)-th element $r_{kl} = 0$ if $k+l$ is even, where the (i,j)-th element of permutation matrix $\widetilde{P}$ is the Kronecker $\delta_{i,s+1-j}$.

In [9] Hairer and Wanner have found that the stability function in terms of the transformed RK-matrix $X = W^{-1}AW$ can be expressed as

$$R(z) = \frac{\det(I - zX + ze_1e_1^T)}{\det(I - zX)}, \tag{1.4}$$

or

$$R(z) = 1 + ze_1^T(I - zX)^{-1}e_1, \tag{1.4$'$}$$

that is, $R(z)$ depends only on X and not on the underlying quadrature formula. Thus, Theorem 1.1 condition b) should be a characterization of linear symplectic Runge-Kutta methods, which is based on the W-transformation of Hairer and Wanner. Note that there exists a difference between the definition of transformation matrices

$$X^* = W^{-1}AW$$

and

$$X = W^TBAW,$$

but it is not essential. The two matrices are related by

$$X = W^TBWX^*. \tag{1.5}$$

In general, X and X^* should possess identical properties. We can obtain at least the following result :

Lemma 1.2. *For the transformation matrices specified by $X^* = W^{-1}AW$ and $X = W^TBAW$ respectively, if one of $(X - \frac{1}{2}e_1e_1^T)$ and $(X^* - \frac{1}{2}e_1e_1^T)$ satisfies condition b) in Theorem 1.1, then the other does also if and only if the (k,l)-th element of matrix W^TBW vanishes if $k+l$ is odd.*

Proof. Let

$$\widetilde{I} = \begin{pmatrix} 1 & 0 & \cdots & \\ 0 & -1 & 0 \cdots & \\ & \ddots & \ddots & 0 \\ & & & (-1)^{s-1} \end{pmatrix}$$