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NUMERICAL ANALYSIS OF NONSTATIONARY THERMISTOR PROBLEM*1)

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Abstract

The thermistor problem is a coupled system of nonlinear PDEs which consists of the heat equation with the Joule heating as a source, and the current conservation equation with temperature dependent electrical conductivity. In this paper we make a numerical analysis of the nonsteady thermistor problem. $L^\infty(\Omega)$, $W^{1,\infty}(\Omega)$ stability and error bounds for a piecewise linear finite element approximation are given.

1. A Mathematical Model and a Discrete Scheme

The model of a nonstationary thermistor problem is derived from the conservation laws of current and energy (see [1] [2] [3]):

Find a pair $\{\varphi, u\}$ such that

$$\nabla \cdot (\sigma(u)\nabla\varphi) = 0 \qquad \text{in} \quad Q_T = \Omega \times (0,T), \tag{1.1}$$

$$\varphi = \varphi_\partial \qquad \text{on} \quad \partial\Omega \times (0,T), \tag{1.2}$$

$$u_t - \Delta u = \sigma(u) \mid \nabla\varphi \mid^2 \quad \text{in} \quad Q_T, \tag{1.3}$$

$$u = 0 \qquad \text{on} \quad \partial\Omega \times (0,T), \tag{1.5}$$

$$u(x,0) = u_0(x) \qquad \text{in} \quad \Omega \tag{1.5}$$

where $\Omega \subset \mathbf{R}^N (N \geq 1)$ is a bounded domain, occupied by the thermistor; $\varphi = \varphi(x,t)$, $u = u(x,t)$ are distributions of the electrical potential and the temperature in Ω, respectively; $\sigma(u)$ is the temperature dependent electrical conductivity; $\sigma(u) \mid \nabla\varphi \mid^2$ is the Joule heating. Throughout this paper, we assume that $0 < \sigma_1 \leq \sigma(s) \leq \sigma_2 < +\infty \ \ \forall s, \in \mathbf{R}^1$.

There has been interest in the problem mathematically (see [1] [2] [3]) and references therein recent mathematical. Yuan [3] proved the following result.

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Theorem 1. *If* $\varphi_\partial \in L^\infty(0,T;C^{1+\alpha}(\bar\Omega))$, $u_0 \in C^\alpha(\bar\Omega)\cap H_0^1(\Omega)$, $0<\alpha<1$, $\sigma(s)\in C^\alpha(\mathbf{R}^1)$, *then problem* (1.1)–(1.5) *has a uniqe solution* (φ,u) *satisfying*

$$u\in C^{\beta,\frac{\beta}{2}}(\bar Q_T),\varphi\in L^\infty(0,T;C^{1+\beta}(\bar\Omega))$$

and

$$\|u\|_{C^{\beta,\frac{\beta}{2}}(\bar Q_T)}\le C,\|\varphi\|_{L^\infty(0,T;C^{1+\beta}(\bar\Omega))}\le C$$

where $\beta\in(0,\alpha)$, *and* C *depends only on the given data.*

As a corollary, we have

Theorem 2. *Under the conditions of Theorem 1 and* $\sigma(s)\in C^1(\mathbf{R}^1)$, $\varphi_\partial\in L^\infty(0,T;H^2(\Omega))$,

(1) *If* $u_0\in H^2(\Omega)\cap H_0^1(\Omega)$, *then*

$$u\in W_P^{2,1}(Q_T),\ \ \forall\ 2\le P<+\infty;\ \ \varphi\in L^\infty(0,T;H^2(\Omega)) \tag{1.6}$$

(2) *If* $u_0\in H^3(\Omega)\cap H_0^1(\Omega)$, *then*

$$u_t\in W_P^{1,0}(Q_T),\ \forall 2\le P<+\infty;\ \ \ u\in L^\infty(0,T;H^2(\Omega)) \tag{1.7}$$

Proof. (1) From Theorem 1, $\sigma(u)\mid\nabla\varphi\mid^2\in L^\infty(Q_T)$. By the standard parabolic estimate[7], (1.3) gives

$$u\in W_P^{2,1}(Q_T),\ \ \ \forall 2\le P<+\infty.$$

Furthermore, by the Corollary in [7]

$$\exists\gamma\in(0,1),\ \ \ u_{x_i}\in C^\gamma(\bar Q_T),\ \ i=1,2,\cdots,N$$

Therefore, $\sigma(u)\in C^0(0,T;C^1(\bar\Omega))$.

By the standard elliptic estimate, from (1.1) we get

$$\varphi\in L^\infty(0,T;H^2(\Omega))$$

(2) From (1.3), we gain

$$u_{x_i,t}-\triangle u_{x_i}=\sigma'(u)u_{x_i}\mid\nabla\varphi\mid^2+2\sigma(u)\nabla\varphi\cdot\nabla\varphi_i\in L^\infty(Q_T).$$

It follows that

$$u_{x_i}\in W_P^{2,1}(Q_T),\ \ \forall\ 2\le P<+\infty. \tag{1.8}$$

Hence, $u_t\in W_P^{1,0}(Q_T)$, $\forall\ 2\le P<+\infty$.

On the other hand, by the embedding theory, we again have

$$\exists\gamma'\in(0,1),u_{x_ix_j}\in C^{\gamma'}(\bar Q_T),\ i,j=1,2,\cdots,N.$$

Now the theorem is proved.

Problem (1.1)–(1.5) has a weak form as follows, Find $u\in H_0^1(\Omega)$, $\varphi\in\varphi_\partial+H_0^1(\Omega)$, such that

$$(\sigma(u)\nabla\varphi,\nabla\psi)=0,\ \ \ t\in(0,T),\ \ \ \forall\psi\in H_0^1(\Omega), \tag{1.9}$$

$$(u_t,v)+(\nabla u,\nabla v)=(\sigma(u)\mid\nabla\varphi\mid^2,v)\ \ \ t\in(0,T),\ \ \ \forall v\in H_0^1(\Omega), \tag{1.10}$$

$$u(x,0)=u_0(x),\ \ \ \ x\in\Omega, \tag{1.11}$$