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A CLASS OF MODIFIED BROYDEN ALGORITHMS*[1)]

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Abstract

In this paper we discuss the convergence of the modified Broyden algorithms. We prove that the algorithms are globally convergent for the continuous differentiable function and the rate of convergence of the algorithms is one-step superlinear and n-step second-order for the uniformly convex objective function. From the discussion of this paper, we may get some convergence properties of the Broyden algorithms.

1. Introduction

We know that the variable metric algorithms, such as the Broyden algorithms, are very useful and efficient methods for solving the nonlinear programming problem

$$min\{f(x); x \in R^n\}. \tag{1.1}$$

With exact linear search, Powell(1971) proves that the rate of convergence of these algorithms is one-step superlinear for the uniformly convex objective function, and if the points given by these algorithms are convergent, Pu and Yu(1990) prove that they are globally convergent for the continuous differentiable function. Without exact linear search several results have been obtained. Powell(1976) demonstrates that the convergence rate of the BFGS algorithms without exact linear search is one-step superlinear. Byrd, Nocedal and Yuan(1987) prove that the above result is also true for other Broyden algorithms except the DFP algorithms. Pu(1990, 1992 and 1993) proves that the convergence rate of the prime DFP algorithms without exact linear search is one-step superlinear for the modified Wolfe conditions.

However there are several theoretical problems which have not been solved for the Broyden algorithms today, and some numerical results show that the points given by the

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Broyden algorithms may not converge to the optimal point for the objective function without convexity(Fletcher(1987)). Several modified variable metric algorithms are proposed for solving those problems and increasing the speed of convergence. In this paper we propose a new class of variable metric algorithms called modified Broyden algorithms which is generalized the idea in Pu's (1989a) short paper and prove the algorithms are convergent for the continuous differentiable objective functions, and superlinear and n-step second order convergent for the uniformly convex functions when the linear search is exact.

The modified Broyden algorithms are iterative. Given a starting point x_1, an initial positive definite matrix B_1 and a constant $\phi \in [0, 1]$, it generates a sequence of points $\{x_k\}$ and a sequence of matrices of $\{B_k\}$, satisfying (1.2) and (1.3)

$$x_{k+1} = x_k + s_k = x_k - \alpha_k B_k^{-1} g_k \tag{1.2}$$

where $\alpha_k \geq 0$ is the step factor, and g_k is the gradient of f(x) at x_k. If $g_k = 0$, the algorithm terminates, otherwise let

$$B_{k+1} = \tilde{B}_{k+1} - \frac{p_{k+1}\tilde{B}_{k+1}R_{k+1}g_{k+1}g_{k+1}^T R_{k+1}\tilde{B}_{k+1}}{1 + p_{k+1}g_{k+1}^T R_{k+1}\tilde{B}_{k+1}R_{k+1}g_{k+1}} \tag{1.3}$$

where p_{k+1} is a positive real number

$$p_{k+1} = \frac{\|Q_{k+1}\tilde{B}_{k+1}^{-1}g_{k+1}\|}{g_{k+1}^T R_{k+1}g_{k+1}}, \tag{1.4}$$

where $\{Q_{k+1}\}$ and $\{R_{k+1}\}$ are two sequences of positive matrices which are uniformly bounded. All eigenvalues of these matrices are included in [q,r], $0 < q \leq r$, And the $\tilde{B}_{k+1}$ is given by

$$\tilde{B}_{k+1} = B_k - \frac{B_k s_k s_k^T B_k}{s_k^T B_k s_k} + \frac{y_k y_k^T}{y_k^T s_k} + \phi(s_k^T B_k s_k) v_k v_k^T, \tag{1.5}$$

where $y_k = g_{k+1} - g_k, v_k = y_k(y_k^T s_k)^{-1} - B_k s_k (s_k^T B_k s_k)^{-1}$. In above programming if B_k are taken $\tilde{B}_k$ for all k, we get the Broyden algorithms. And if $\phi = 0$ we call it modified BFGS algorithms, or abbreviated by MBFGS and if $\phi = 1$ we call it modified DFP algorithms, or MDFP algorithms for short.

The matrix H_k and $\tilde{H}_k$ are denoted the inverses of B_k and $\tilde{B}_k$, we may obtain the Quasi-Newton formula $\tilde{H}_{k+1}y_k = s_k$ by the Broyden algorithms. And Pu(1989a) gave

$$H_{k+1} = \tilde{H}_{k+1} + p_{k+1}R_{k+1}g_{k+1}g_{k+1}^T R_{k+1}. \tag{1.6}$$

From the Broyden algorithms we know that if H_k is positive definitive the $\tilde{H}_{k+1}$ is also positive definitive. H_{k+1} may be implied positive. To use the mathematical induction it is easy to imply the $\tilde{B}_{k+1}, B_{k+1}, \tilde{H}_{k+1}$ and H_{k+1} are positive definitive matrices by the H_1 and B_1 being.