Journal of Computational Mathematics
Vol.38, No.3, 2020, 417–436.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1811-m2016-0763

IMAGE DENOISING VIA TIME-DELAY REGULARIZATION COUPLED NONLINEAR DIFFUSION EQUATIONS*

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Abstract

A novel nonlinear anisotropic diffusion model is proposed for image denoising which can be viewed as a novel regularized model that preserves the cherished features of Perona-Malik to some extent. It is characterized by a local dependence in the diffusivity which manifests itself through the presence of $p(x)$-Laplacian and time-delay regularization. The proposed model offers a new nonlinear anisotropic diffusion which makes it possible to effectively enhance the denoising capability and preserve the details while avoiding artifacts. Accordingly, the restored image is very clear and becomes more distinguishable. By Galerkin's method, we establish the well-posedness in the weak setting. Numerical experiments illustrate that the proposed model appears to be overwhelmingly competitive in restoring the images corrupted by Gaussian noise.

Mathematics subject classification: 65N06, 65B99.
Key words: Image denoising, Galerkin's method, Existence, Uniqueness.

1. Introduction

Image denoising is the paramount concept in the domain of image processing. The literature about this topic is vast and many methods have been devised, which perform the task to various degrees of satisfaction. During the last decade, linear diffusion filters [12, 16, 17, 20, 31], nonlinear diffusion filters [1, 5, 7, 8, 14, 23, 34] and energy norm minimization-based variational methods [2–4, 9, 22, 24, 25, 28] have become powerful and well-founded tools in image denoising.

In a well-known paper, Perona and Malik [23] developed an adaptive smoothing and edge detection scheme in which the linear heat diffusion equation is replaced by a selective diffusion that preserves edges. It reads

$$\frac{\partial u}{\partial t} = \operatorname{div}\big(g\big(|\nabla u|\big)\nabla u\big), \tag{1.1}$$

where

$$g(s) = \frac{1}{1+(s/\alpha)^2}, \qquad \alpha > 0.$$

The Perona-Malik equation is well posed only if $sg(s)$ is a nondecreasing function [7]. However, it can be successfully implemented to give a viable denoising tool for image processing with remarkable preservation of sharp edges. It is particularly in the matter of this discrepancy

* Received October 10, 2016 / Revised version received June 14, 2018 / Accepted November 22, 2018 / Published online February 11, 2019 /

between its mathematical properties and its practical stability and efficacy that the mathematical community has shown an unfading interest in the Perona-Malik equation since it was first introduced in [23]. The technique was improved by many other contributions [1,5–7,21] which could be viewed as the paradigm for the regularization of the well-known Perona-Malik equation. However, these methods tend to introduce a certain amount of blurring. Later, Nitzberg and Shiota [21] proposed the time-delay regularization which could adjust over-smoothing. A similar equation involving the time-delay regularization of an anisotropic diffusion tensor has been already studied, also for image processing, by Cottet and EI Ayyadi [6].

Since fractional derivatives based models have been studied in computer vision, many methods in connecting fractional derivatives with systems of nonlinear partial differential equations or variational methods have been proposed to preserve important structures in images, while removing noise. For example, Guidotti [14] proposed a nonlocal nonlinear diffusion for the regularization of the well-known Perona-Malik equation which is implemented via nonlinearity intensity reduction through fractional derivatives. Dong and Chen [10] proposed a unified variational framework for noise removal using the fractional-order derivatives. But the selection of the fractional orders is not automatic. In our previous work [19], a fractional differential fidelity-based PDE model which can be viewed as a denoising tool was proposed. The method suggests that for denoising, it is reasonable and practicable to prevent the staircase effect and simultaneously enhance the noisy image using the combination of fractional-order fidelity term and global fidelity term.

In this paper, we propose a new methodology for image denoising by incorporating time-delay regularization into $p(x)$-Laplacian based anisotropic diffusion. Specifically, the smoothing strength can be adaptively modulated with respect to the value of $p(x)$ at each location, and the time-delay regularization is also used to adjust the smoothing strength by a weighted time average of the filter process information obtained from images at each step of the iteration process. The time-delay regularization is embedded into $p(x)$-Laplacian based anisotropic diffusion, which allows for a selective smoothing that can effectively enhance the denoising capability and preserve details while avoiding artifacts. More specifically, the restored image is very clear and becomes more distinguishable. Also, comparisons with other denoising results show that the proposed model gives promising denoising results.

The rest of this paper is organized as follows. In Section 2, our denoising model is presented. Section 3 recalls the indispensable theoretical background of Sobolev space with variable exponent and drives the approximate solutions to the proposed model. Well-posedness in the weak setting is established in Section 4. In Section 5, we give the detailed implementation of our method. In Section 6, we provide experimental results on real-world images. Conclusion remarks are given in Section 7.

2. The Proposed Model

In this section, the proposed model is presented using a kind of $p(x)$-Laplacian based anisotropic diffusion with the time-delay regularization, where each step of the iteration process and each location of the smoothed image u are taken into account by modeling the diffusion term in both explicit and implicit ways. In the following, we describe our model (2.1)-(2.6).

Let Ω be a bounded domain of $\mathbb{R}^N$, $T > 0$, $\lambda > 0$, f is the original image, we restrict the range of f to $[0, 1]$. k_1 and k_2 are positive constants, and $\vec{n}$ denotes the unit outward normal to the boundary $\partial\Omega$.