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CONVERGENCE OF LAPLACIAN SPECTRA FROM RANDOM SAMPLES*

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Abstract

Eigenvectors and eigenvalues of discrete Laplacians are often used for manifold learning and nonlinear dimensionality reduction. Graph Laplacian is one widely used discrete laplacian on point cloud. It was previously proved by Belkin and Niyogithat the eigenvectors and eigenvalues of the graph Laplacian converge to the eigenfunctions and eigenvalues of the Laplace-Beltrami operator of the manifold in the limit of infinitely many data points sampled independently from the uniform distribution over the manifold. Recently, we introduced Point Integral method (PIM) to solve elliptic equations and corresponding eigenvalue problem on point clouds. In this paper, we prove that the eigenvectors and eigenvalues obtained by PIM converge in the limit of infinitely many random samples. Moreover, estimation of the convergence rate is also given.

Mathematics subject classification: 62G20, 65N25, 60D05.
Key words: Graph Laplacian, Laplacian spectra, Random samples, Spectral convergence.

1. Introduction

In the past decade, data science plays more and more important role in sciences, engineering and our daily lives. Among varieties of data analysis methods and models, manifold model attracts more and more attentions. In the manifold model, data is represented as a point cloud, which is defined as a collection of points that are embedded in a high dimensional Euclidean space. It is assumed that the point cloud samples a smooth manifold. Thus, the structure of the manifold are very useful to understand the data. On the other hand, research in mathematics shows that the Laplace-Beltrami operator is one of the most important object associated to Riemannian manifolds. Its eigenvalue and eigenfunctions encode all intrinsic geometry of the manifolds. To reveal the structure of the underlying manifold sampled by the data, many discrete counterparts of LBO are developed. The eigenvalues and eigenvectors of the discrete Laplace-Beltrami operators are widely used in many fields, including machine learning, data analysis, computer graphics and computer vision, and geometric modeling and processing [2, 6, 16, 18]. Then, one question is that if the eigenvalues and eigenvectors of these discrete operators converge to the eigenvalues and eigenfunctions of their continuous counterpart, Laplace-Beltrami operator as the point cloud converges to the manifold. This is essential to understand these discrete operators and algorithms associated to them.

The convergence between the graph Laplacian and the Laplace-Beltrami operator has been studied extensively in the literature [3, 4, 7–9, 13, 21, 23]. In the presence of no boundary and

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the sample points are uniformly distributed, Belkin and Niyogi [4] showed that the spectra of the normalized graph Laplacian converges to the spectra of Laplace-Beltrami operator. When there is a boundary, it was observed in [5] and [13] that the integral Laplace operator L_t is dominated by the first order derivative and thus fails to be true Laplacian near the boundary. Recently, Singer and Wu [22] showed the spectral convergence in the presence of the Neumann boundary. In this paper, we study this problem from another point of view. We study the solution operators of graph Laplacian and Laplace-Beltrami operator. Based on the convergence between the solutions operators, we get more delicate estimate of the convergence, include the convergence rate.

In this paper, we assume that the data points, $X_n = \{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$, are sampled independently over the manifold $\mathcal{M}$ from a probability distribution $p(\mathbf{x})$. On the sample points, we consider following discrete eigenvalue problem.

$$\frac{1}{t}\sum_{j=1}^{n} R\left(\frac{\|\mathbf{x}_i-\mathbf{x}_j\|^2}{4t}\right)(u_i-u_j) = \lambda\sum_{j=1}^{n}\bar{R}\left(\frac{\|\mathbf{x}_i-\mathbf{x}_j\|^2}{4t}\right)u_j, \tag{1.1}$$

where $R : \mathbb{R}^+ \to \mathbb{R}^+$ is a kernel function satisfies some conditions (see Assumption 3.1), $\bar{R}(r) = \int_r^{+\infty} R(s)\mathrm{d}s$.

The purpose of this paper is to study the behavior of discrete eigenvalue problem (1.1) as $n \to \infty$ and $t \to 0$. We show that when $n \to \infty$ and $t \to 0$, the spectral of (1.1) converge to the spectra of the following Laplace-Beltrami operator,

$$\begin{cases} -\frac{1}{p^2(\mathbf{x})}\operatorname{div}\left(p^2(\mathbf{x})\nabla u(\mathbf{x})\right) = \lambda u(\mathbf{x}), & \mathbf{x}\in\mathcal{M}, \\ \frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) = 0, & \mathbf{x}\in\partial\mathcal{M}. \end{cases} \tag{1.2}$$

where $\mathbf{n}$ is the out normal vector of $\mathcal{M}$.

Remark 1.1. The eigenvalue problem we consider here is a little different as the traditional graph Laplacian. Graph Laplacian L is given by $L = \boldsymbol{I} - \boldsymbol{D}^{-1}\boldsymbol{W}$. Here weight matrix W has expression $W_{i,j} = R(\|\boldsymbol{x_i} - \boldsymbol{x_j}\|^2/4t)$, D is a diagonal matrix whose elements are the row sums of $\boldsymbol{W}$ and $\boldsymbol{I}$ is the identity matrix. In traditional graph Laplacian framwork, the discrete eigenvalue problem is

$$\frac{1}{t}\sum_{j=1}^{n} R\left(\frac{\|\mathbf{x}_i-\mathbf{x}_j\|^2}{4t}\right)(u_i-u_j) = \lambda u_i\sum_{j=1}^{n} R\left(\frac{\|\mathbf{x}_i-\mathbf{x}_j\|^2}{4t}\right),$$

which is different from (1.1) in the right hand side.

Tha analysis in this paper is based on the point integral method [15] The main idea of the point integral method is to approximate the Poisson equation via an integral equation:

$$\begin{aligned} &-\int_{\mathcal{M}}\Delta_{\mathcal{M}}u(\mathbf{y})\bar{R}_t(\mathbf{x},\mathbf{y})d\mu_{\mathbf{y}} \\ \approx&\frac{1}{t}\int_{\mathcal{M}}R_t(\mathbf{x},\mathbf{y})(u(\mathbf{x})-u(\mathbf{y}))\mathrm{d}\mu_{\mathbf{y}} - 2\int_{\partial\mathcal{M}}\bar{R}_t(\mathbf{x},\mathbf{y})\frac{\partial u}{\partial \mathbf{n}}(\mathbf{y})\mathrm{d}\tau_{\mathbf{y}}, \end{aligned} \tag{1.3}$$

where $\mathbf{n}$ is the out normal of $\partial\mathcal{M}$, $\mathcal{M}$ is a smooth k-dimensional manifold embedded in $\mathbb{R}^d$, $\partial\mathcal{M}$ is the boundary of $\mathcal{M}$. $R_t(\mathbf{x},\mathbf{y})$ and $\bar{R}_t(\mathbf{x},\mathbf{y})$ are kernel functions same as those in (1.1). $\Delta_{\mathcal{M}} = \operatorname{div}(\nabla)$ is the Laplace-Beltrami operator (LBO) on $\mathcal{M}$.