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BOUNDARY VALUE METHODS FOR CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS*

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Abstract

This paper deals with the numerical computation and analysis for Caputo fractional differential equations (CFDEs). By combining the p-order boundary value methods (B-VMs) and the m-th Lagrange interpolation, a type of extended BVMs for the CFDEs with γ-order $(0<\gamma<1)$ Caputo derivatives are derived. The local stability, unique solvability and convergence of the methods are studied. It is proved under the suitable conditions that the convergence order of the numerical solutions can arrive at $\min\{p, m-\gamma+1\}$. In the end, by performing several numerical examples, the computational efficiency, accuracy and comparability of the methods are further illustrated.

Mathematics subject classification: 34A08, 34K28, 65L20.
Key words: Fractional differential equations, Caputo derivatives, Boundary value methods, Local stability, Unique solvability, Convergence.

1. Introduction

In this paper, we consider the following initial value problems of CFDEs

$$y'(t)=f\left(t, y(t), {}_{t_0}^{C}D_t^{\gamma}y(t)\right), \quad t\in[t_0,T]; \quad y(t_0)=y_0, \tag{1.1}$$

where $f:[t_0,T]\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}^d$ is a given sufficiently smooth function, $y_0\in\mathbb{R}^d$ is an assigned initial value and ${}_{t_0}^{C}D_t^{\gamma}y(t)$ is the γ-order Caputo derivative of the unknown function $y(t)$ defined by (cf. [30, 32, 36])

$${}_{t_0}^{C}D_t^{\gamma}y(t)=\frac{1}{\Gamma(1-\gamma)}\int_{t_0}^{t}\frac{y'(v)}{(t-v)^{\gamma}}dv, \quad 0<\gamma<1. \tag{1.2}$$

The model (1.1) has a wide application in science and technology. For example, in McKee [28] and McKee & Stokes [29], the diffusion of discrete particles in a turbulent fluid is modeled by the so-called Basset equation:

$$y'(t)=f(t,y(t))+c(t)\int_{t_0}^{t}\frac{y'(v)}{(t-v)^{\gamma}}dv+g(t), \quad t\in[t_0,T]; \quad y(t_0)=y_0, \tag{1.3}$$

where $f(t,y(t))$, $c(t)$ and $g(t)$ are the assigned functions. An extended Basset equation

$$y'(t)=f(t,y(t))+\frac{1}{\Gamma(1-\gamma)}\int_{t_0}^{t}\frac{k(t,v,y'(v))}{(t-v)^{\gamma}}dv, \quad t\in[t_0,T]; \quad y(t_0)=y_0, \tag{1.4}$$

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can be found in Brunner & Tang [7] and Hairer & Maass [14]. Another example is the Babenko's model describing the gas pressure in a fluid (cf. [3]) which is given by

$$\begin{cases} \dfrac{\partial}{\partial t}\left[V_0\ g(t/\theta)\ P(t,0)\dfrac{M}{RT}\right] = F\mathcal{D}\ \dfrac{\partial C}{\partial x}\Big|_{x=0}, \\ -\sqrt{\mathcal{D}}\ \dfrac{\partial C}{\partial x}\Big|_{x=0} = {}_0^C D_t^{1/2}[C(t,0)-C(0,x)], \quad t\in[0,\theta], \\ P(t,0)=\kappa C(t,0), \quad P(0,x)=\kappa C(0,x), \quad x\in[0,\infty), \end{cases} \tag{1.5}$$

where V_0 is the initial gas volume, θ is the time of the gas compression to zero volume, $g(t/\theta)$ is the function reflecting the change of gas volume with $g(0)=1$ and $g(1)=0$, $M, R, \mathcal{D}, F$ denote the gas molar weight, universal gas constant, diffusion coefficient of gas in the fluid and contact surface between the gas and the fluid, respectively, κ is the Henry's constant, $C(t,x)$ is the gas concentration and $P(t,x)$ is the unknown gas pressure. The gas temperature T is assumed to be constant. From the problem (1.5), we can obtain the following initial-value problem for determining the dimensionless gas pressure $p(t)\equiv p(t,x)=\frac{P(t,x)}{P(0,x)}$ near the contact surface:

$$\frac{d}{dt}(g(t)p(t)) + \lambda\, {}_0^C D_t^{1/2}[p(t)-1] = 0, \quad t\in[0,1]; \quad p(0)=1. \tag{1.6}$$

Let $y(t)=p(t)-1, G(t)=g(t)/g'(t)$ and $\hat{G}(t)=\lambda/g'(t)$. Then (1.6) can be written as a FDE of the form (1.1):

$$G(t)y'(t)+\hat{G}(t)\, {}_0^C D_t^{1/2}y(t)+y(t)=-1, \quad t\in[0,1]; \quad y(0)=0.$$

Besides the above real models, with the semi-discrete method for the spatial variable x, which is also called *method of lines*, the following fractal mobile/immobile transport models (cf. [26,33]):

$$\begin{cases} a_1\dfrac{\partial u(x,t)}{\partial t} + a_2\, {}_0^C D_t^{\gamma} u(x,t) = a_3\dfrac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad (x,t)\in[a,b]\times[t_0,T], \\ u(x,0)=\varphi_0(x), \quad x\in[a,b], \\ u(a,t)=\phi_1(t), \quad u(b,t)=\phi_2(t), \quad t\in[t_0,T] \end{cases}$$

can be transformed into (1.1). A detailed description for this approach refers to Example 6.2. Moreover, some other fractional partial differential equations, such as fractional reaction-subdiffusion equation (cf. [21, 24]), fractional cable equation (cf. [25]) and the equations in references [30, 32, 36], can also be cast into (1.1) by the method of lines.

In contrast to the classical regular Volterra integro-differential equations, the CFDEs have the weakly singular factor $(t-v)^{-\gamma}$ $(0<\gamma<1)$, which leads to the difficulties to obtain the solutions of the equations. Hence, developing various numerical methods for CFDEs becomes an important issue. In [28,29], for Basset equation (1.3), McKee and Stokes proposed the product integration methods based on backward difference interpolation. Subsequently, for the extended Basset equations (1.4), Brunner and Tang [7] constructed the polynomial spline collocation methods and Hairer and Maass [14] presented the fractional linear multistep methods. As to the other related researches for CFDEs, the readers can find them in [22, 26, 30, 32, 36] and the references therein. It should be pointed out that, most of the existed numerical methods for CFDEs are presented for the regularity problems (see e.g. [8, 22, 27]). However, in general, the solutions of problems (1.1) have the weak singularity at initial point. Hence, it is necessary to consider some computational techniques to treat this issue in order to obtain the expected