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SUPERCONVERGENCE ANALYSIS OF LOW ORDER NONCONFORMING MIXED FINITE ELEMENT METHODS FOR TIME-DEPENDENT NAVIER-STOKES EQUATIONS*

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Abstract

In this paper, the superconvergence properties of the time-dependent Navier-Stokes equations are investigated by a low order nonconforming mixed finite element method (MFEM). In terms of the integral identity technique, the superclose error estimates for both the velocity in broken H^1-norm and the pressure in L^2-norm are first obtained, which play a key role to bound the numerical solution in L^∞-norm. Then the corresponding global superconvergence results are derived through a suitable interpolation postprocessing approach. Finally, some numerical results are provided to demonstrated the theoretical analysis.

Mathematics subject classification: 65N38, 65N30, 65M60, 65M12.
Key words: Navier-Stokes equations, Nonconforming MFEM, Supercloseness and superconvergence.

1. Introduction

In this paper, we focus on the following time-dependent Navier-Stokes equations in 2D:

$$\boldsymbol{u}_t - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f}, \qquad (\boldsymbol{x},t) \in \Omega\times(0,T], \tag{1.1}$$

$$\nabla\cdot\boldsymbol{u} = 0, \qquad (\boldsymbol{x},t) \in \Omega\times(0,T], \tag{1.2}$$

$$\boldsymbol{u}(\boldsymbol{x},t) = 0, \qquad (\boldsymbol{x},t) \in \partial\Omega\times(0,T], \tag{1.3}$$

$$\boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}), \qquad \boldsymbol{x}\in\Omega, \tag{1.4}$$

where $\Omega\subset\mathbb{R}^2$ is a rectangular domain with boundary $\partial\Omega$ and $\boldsymbol{x} = (x_1,x_2)$. $\boldsymbol{u} = (u_1,u_2)$ represents the velocity vector, p the pressure, $\boldsymbol{f} = (f_1,f_2)$ the body force, $\nu = 1/Re$ the viscosity coefficient and Re is the Reynolds number.

It is well known that the incompressible Navier-Stokes equations are of great importance both in mathematics and fluid mechanics. There have been a large number of works concentrated on the numerical solutions of Navier-Stokes equations. We refer the readers to monographs [1,2] for the theoretical and numerical analysis, [3–6] for finite difference methods, [7–24] for FEMs, [25–27] for characteristics FEMs, [28, 29] for discontinuous Galerkin method. More

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precisely, ax fast finite difference method was proposed in [4] based on the vorticity stream-function formulation. A backward Euler fully-discrete penalty FEM was presented in [7] and an optimal error estimate was provided when the corresponding parameters were sufficiently small. Through the spatial discretization by finite element approximation and the time discrtization by the semi-implicit scheme, a fully-discrete stabilized FEM was studied in [8]. In addition, a stabilized FEM was considered by use of the local polynomial pressure projection with the lowest equal-order elements in [9]. The two-level finite element Galerkin method was employed to deduce the corresponding optimal error estimates in [10] and [11], respectively. Moreover, a class of nonconforming rectangular elements were used in [16] and an optimal estimate was obtained. Two kinds second order nonconforming mixed FEMs were developed and optimal error estimates were derived in [22] and [23], respectively. In [27], the unconditional stability and convergence of characteristics type method was studied and an optimal error estimate was achieved.

As far as we know, all of the above works are concerned with convergence analysis and optimal error estimates. Recently, the superconvergnece analysis was researched with non-conforming mixed FEM (CNRQ_1+Q_0, see the Section 2 for the definition) for the stationary Navier-Stokes equations and time-dependent Navier-Stokes equations in [30] and [31], respectively. However, only the error estimate for the spatial semi-discrete scheme was considered in [31] and the error estimate is not valid when $t \to 0$.

In this paper, we will focus on the superconvergence analysis for (1.1)-(1.4) by a linearized fully-discrete scheme, in which the spatial discretization approximated by the low order CNRQ_1 element (cf. [32, 33]) for the velocity, and the piecewise constant for the pressure and the time discretization approximated by the semi-implicit Euler scheme. It should be mentioned that the factor $1/t$ required in [31] is removed in our present work, which shows that the error estimates are also valid when $t \to 0$.

The rest of this paper is organized as follows. In Section 2, we briefly introduce the non-conforming finite element spaces and some lemmas. In Section 3, we discuss the superclose and superconvergence analysis for (1.1)-(1.4). In the last section, we carry out two numerical experiments to confirm the theoretical analysis.

2. The Finite Element Spaces and Some Lemmas

We will use the standard notations for the Sobolev space $H^m(\Omega),\ m \geq 0$ (cf. [34]) with their associated norm $\|\cdot\|_m$ and seminorm $|\cdot|_m$. In the case $m = 0$, then $H^0(\Omega) = L^2(\Omega)$, the norm and inner product are denoted by $\|\cdot\|_0$ and $(\cdot,\cdot)$, respectively. We let $L_0^2(\Omega)$ denote the subspace of $L^2(\Omega)$ such that

$$L_0^2(\Omega) = \Big\{ v \in L^2(\Omega) \, : \, \int_\Omega v dx_1 dx_2 = 0 \Big\}.$$

In addition, for any Banach space X and $I = [0, T]$, let $L^p(I; X)$ be the space of all measurable function $f : I \to X$ with the norm

$$\|f\|_{L^p(I;X)} = \begin{cases} (\int_0^T \|f\|_X^p dt)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \mathrm{esssup}_{t\in I} \|f\|_X, & p = \infty. \end{cases}$$