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WELL-CONDITIONED FRAMES FOR HIGH ORDER FINITE ELEMENT METHODS*

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Abstract

The purpose of this paper is to discuss representations of high order C^0 finite element spaces on simplicial meshes in any dimension. When computing with high order piecewise polynomials the conditioning of the basis is likely to be important. The main result of this paper is a construction of representations by frames such that the associated L^2 condition number is bounded independently of the polynomial degree. To our knowledge, such a representation has not been presented earlier. The main tools we will use for the construction is the bubble transform, introduced previously in [1], and properties of Jacobi polynomials on simplexes in higher dimensions. We also include a brief discussion of preconditioned iterative methods for the finite element systems in the setting of representations by frames.

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Key words: Finite element method, High order, Condition number, Frame, Preconditioner.

1. Introduction

The discussion in this paper is motivated by finite element discretizations of second order elliptic equations, where C^0 piecewise polynomial spaces of high polynomial degree are used as the finite dimensional space. As the polynomial degree increases the choice of basis can have a substantial effect on the conditioning of the linear systems to be solved. The purpose of this paper is to discuss how to obtain representations of the finite element spaces which are uniformly well-conditioned with respect to the polynomial degree. Here the conditioning of the representation is measured by the L^2 condition number. Furthermore, we will explain how this influences the conditioning of the corresponding discrete systems. Since our main goal is to discuss dependence with respect to the polynomial degree we will consider the mesh $\mathcal{T}_h$ to be fixed throughout the discussion below.

To motivate the discussion below, we consider a second order elliptic equation, defined on a bounded domain $\Omega \in \mathbb{R}^d$, which admits a weak formulation of the form:

Find $u \in H^1(\Omega)$ such that

$$a(u, v) = f(v), \quad v \in H^1(\Omega), \tag{1.1}$$

where $H^1(\Omega)$ denotes the Sobolev space of all functions in L^2 which also have all first order partial derivates in L^2. Furthermore, f is a bounded linear functional, and a is a symmetric,

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bounded, and coercive bilinear form on $H^1(\Omega)$. The formulation above reflects that we are considering an elliptic problem with natural boundary condition. If we instead consider problems with an essential boundary condition on parts of the boundary, we will obtain a weak formulation with respect to a corresponding subspace of $H^1(\Omega)$. However, the effect of such modifications of (1.1) will have minor effects on the discussion below. Therefore, we will restrict the discussion to problems of the form (1.1) throughout this paper.

A discretization of the problem (1.1) can be derived from a finite dimensional subspace V_h of $H^1(\Omega)$. In the finite element method V_h is typically a space of piecewise polynomials with respect to a partition, or a mesh, $\mathcal{T}_h$, with global C^0 continuity, and where the mesh parameter h indicates the size of the cells of the partition. The corresponding discrete solution is defined by:

Find $u_h \in V_h$ such that

$$a(u_h, v) = f(v), \quad v \in V_h. \tag{1.2}$$

This system can alternatively be written as a linear system of the form $\mathcal{A}_h u_h = f_h$, where $f_h \in V_h^*$, and where the operator $\mathcal{A}_h : V_h \to V_h^*$ is defined by $\mathcal{A}_h u(v) = a(u, v)$, for all $u, v \in V_h$. Hence, $\mathcal{A}_h$ is symmetric in the sense that for all $u, v \in V_h$, $\langle \mathcal{A}_h u, v\rangle = \langle \mathcal{A}_h v, u\rangle$, where $\langle \cdot, \cdot \rangle$ is the duality pairing between V_h^* and V_h. To turn the discrete system (1.2) into a system of linear equations, written in a matrix/vector form, we need to introduce a basis $\{\phi_j\}_{j=1}^n$ for the space V_h. This means that any element $v \in V_h$ can be written uniquely on the form $v = \sum_j c_j \phi_j$. We denote the map from $\mathbb{R}^n$ to V_h given by $c \mapsto v$ for τ_h. In a corresponding manner we define $\mu_h : V_h^* \to \mathbb{R}^n$ by $(\mu_h f)_i = \langle f, \phi_i \rangle$. We note that if $f \in V_h^*$ and $c \in \mathbb{R}^n$ then

$$\mu_h f \cdot c = \sum_{i=1}^n \langle f, \phi_i \rangle c_i = \langle f, \tau_h(c)\rangle,$$

where $\mathbb{R}^n$ is equipped with the standard Euclidean inner product, and where we adopt the standard "dot notation" for this inner product. Hence, $\mu_h : V_h^* \to \mathbb{R}^n$ can be identified as τ_h^*. If c is the unknown vector, $c = \tau_h^{-1} u_h$, then the system (1.2) is equivalent to the linear system

$$\mathbb{A}_h c = \mu_h(f_h) \equiv \tau_h^*(f_h), \tag{1.3}$$

where $\mathbb{A}_h$ corresponds to the $n \times n$ matrix representing the operator $\tau_h^* \mathcal{A}_h \tau_h : \mathbb{R}^n \to \mathbb{R}^n$. The matrix $\mathbb{A}_h$ is usually referred to as the stiffness matrix, and the element $(\mathbb{A}_h)_{i,j}$ is given as $a(\phi_i, \phi_j)$. Furthermore, we note that the diagram

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\mathbb{A}_h} & \mathbb{R}^n \\ \tau_h \big\downarrow & & \big\uparrow \tau_h^* \\ V_h & \xrightarrow{\mathcal{A}_h} & V_h^* \end{array} \tag{1.4}$$

commutes. However, there is a striking difference between the operator $\mathcal{A}_h : V_h \to V_h^*$ and its matrix representation $\mathbb{A}_h$. The stiffness matrix $\mathbb{A}_h$ depends strongly on the choice of basis, while the operator $\mathcal{A}_h$ only depends on the bilinear form a and the space V_h.

For piecewise polynomial spaces of high order the choice of basis can have dramatic effect on the conditioning of the stiffness matrix $\mathbb{A}_h$. Therefore, there are a number of contributions in the literature discussing how to choose proper bases for C^0 piecewise polynomial spaces of high