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# AN ADAPTIVE TRUST-REGION METHOD FOR GENERALIZED EIGENVALUES OF SYMMETRIC TENSORS*

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**Abstract**

For symmetric tensors, computing generalized eigenvalues is equivalent to a homogenous polynomial optimization over the unit sphere. In this paper, we present an adaptive trust-region method for generalized eigenvalues of symmetric tensors. One of the features is that the trust-region radius is automatically updated by the adaptive technique to improve the algorithm performance. The other one is that a projection scheme is used to ensure the feasibility of all iteratives. Global convergence and local quadratic convergence of our algorithm are established, respectively. The preliminary numerical results show the efficiency of the proposed algorithm.

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*Key words:* Symmetric tensors, Generalized eigenvalues, Trust-region, Global convergence, Local quadratic convergence.

## 1. Introduction

In 2005, Lim [1] and Qi [2] introduced eigenvalues and eigenvectors of symmetric tensors, independently. Since then, various types of eigenvalues problem for symmetric tensors have received much attention, and a series of promising research results have been obtained [3–7]. In the last decade, tensor eigenproblem has many applications in multilinear algebra [4, 6]. The corresponding results were widely applied in image analysis [8], data fitting [9, 10], signal processing [11], quantum physics [12], automatic control [13], independent component analysis [14, 15], etc.

Consider an $m$th-order $n$-dimensional tensor $\mathcal{A}$

$$\mathcal{A} = (a_{i_1\ldots i_m}), \quad a_{i_1\ldots i_m} \in R, \ \ 1 \le i_1, \ldots, i_m \le n.$$

For any vector $x = (x_1, \ldots, x_n)^{\mathrm{T}} \in R^n$, we denote

$$(\mathcal{A}x^{m-1})_i := \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m} x_{i_2} \cdots x_{i_m}, \qquad \text{for } i = 1, \ldots, n,$$

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$$\mathcal{A}x^m := \sum_{i_1,\ldots,i_m=1}^{n} a_{i_1 i_2 \ldots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}.$$

It is clear that $\mathcal{A}x^{m-1}$ is an $n$-dimensional column vector, $\mathcal{A}x^m$ is an $m$ degree homogenous polynomial, and $\mathcal{A}x^m = x^{\mathrm{T}}\mathcal{A}x^{m-1}$. $\mathcal{A}$ is called positive definite if $\mathcal{A}x^m > 0$ for all $x \neq 0$.

We say $\mathcal{A}$ is symmetric if its entries $a_{i_1 \ldots i_m}$ are invariant under any permutation of their indices $i_1, \ldots, i_m$ [2,16,17]. If $\mathcal{A}$ is symmetric, $\mathcal{A}x^m$ satisfies $\nabla(\mathcal{A}x^m) = m\mathcal{A}x^{m-1}$ [3]. Denote the set of all real symmetric $m$th-order $n$-dimensional tensors by $\mathcal{S}^{[m,n]}$. All tensors considered in this paper, unless stated otherwise, are symmetric.

For a tensor $\mathcal{A}$, if there exist $\lambda \in R$ and $x \in R^n \backslash \{0\}$ satisfying

$$\begin{aligned} \mathcal{A}x^{m-1} &= \lambda x, \\ x^{\mathrm{T}}x &= 1, \end{aligned} \tag{1.1}$$

then $\lambda$ is called a Z-eigenvalue, $x$ is called the corresponding Z-eigenvector, and $(\lambda, x)$ is called a Z-eigenpair of $\mathcal{A}$.

By the variational principle, any vector $x$ satisfying (1.1) is a KKT point of the following polynomial optimization problem

$$\begin{aligned} \max_{x \in R^n} \quad & \mathcal{A}x^m, \\ \text{s.t.} \quad & x^{\mathrm{T}}x = 1, \end{aligned} \tag{1.2}$$

where $(\mathcal{A}x^m, x)$ is the Z-eigenpair of $\mathcal{A}$.

Recently, there were several algorithms to find Z-eigenvalues for solving (1.2). Hao et. al. [18] put forth a sequential subspace projection method for solving Z-eigenvalues of large-scale symmetric tensors. Hao et. al. [19] presented a feasible trust-region method (FTR) for calculating Z-eigenvalues of symmetric tensors. Qi et. al. [20] proposed a direct Z-eigenvalue method when the dimension is two. Then based on orthogonal transformations, they proposed a direct orthogonal transformation Z-eigenvalue method in the case of order three and dimension three.

Let $\mathcal{A}, \mathcal{B} \in \mathcal{S}^{[m,n]}$. Assume that $m$ is even and $\mathcal{B}$ is positive definite, $(\lambda, x) \in R \times R^n \backslash \{0\}$ is called a generalized eigenpair of $(\mathcal{A}, \mathcal{B})$ if it satisfies

$$\mathcal{A}x^{m-1} = \lambda \mathcal{B}x^{m-1}, \tag{1.3}$$

where $\lambda$ is called a generalized eigenvalue, $x$ is called the generalized vector.

One of the practical methods is to transform (1.3) into the following constrained optimization problem

$$\begin{aligned} \max_{x \in R^n} \quad & \mathcal{A}x^m, \\ \text{s.t.} \quad & \mathcal{B}x^m = 1. \end{aligned} \tag{1.4}$$

In [1], Lim showed that the KKT point of (1.4) gives generalized eigenvalues of (1.3).

Another alternative approach for computing generalized eigenpairs (1.3) is to solve the following nonlinear programming problem

$$\begin{aligned} \max_{x \in R^n} \quad & f(x) = \tfrac{\mathcal{A}x^m}{\mathcal{B}x^m}, \\ \text{s.t.} \quad & x^T x = 1. \end{aligned} \tag{1.5}$$