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# A MIXED VIRTUAL ELEMENT METHOD FOR THE BOUSSINESQ PROBLEM ON POLYGONAL MESHES*

Gabriel N. Gatica[1) ] and Mauricio Munar
*CI²MA and Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile*
*Email: ggatica@ci2ma.udec.cl, mmunar@ci2ma.udec.cl*
Filánder A. Sequeira
*Escuela de Matemática, Universidad Nacional, Campus Omar Dengo, Heredia, Costa Rica*
*Email: filander.sequeira@una.cr*

**Abstract**

In this work we introduce and analyze a mixed virtual element method (mixed-VEM) for the two-dimensional stationary Boussinesq problem. The continuous formulation is based on the introduction of a pseudostress tensor depending nonlinearly on the velocity, which allows to obtain an equivalent model in which the main unknowns are given by the aforementioned pseudostress tensor, the velocity and the temperature, whereas the pressure is computed via a postprocessing formula. In addition, an augmented approach together with a fixed point strategy is used to analyze the well-posedness of the resulting continuous formulation. Regarding the discrete problem, we follow the approach employed in a previous work dealing with the Navier-Stokes equations, and couple it with a VEM for the convection-diffusion equation modelling the temperature. More precisely, we use a mixed-VEM for the scheme associated with the fluid equations in such a way that the pseudostress and the velocity are approximated on virtual element subspaces of $\mathbb{H}(\mathrm{div})$ and $\mathbf{H}^1$, respectively, whereas a VEM is proposed to approximate the temperature on a virtual element subspace of $\mathrm{H}^1$. In this way, we make use of the $\mathrm{L}^2$-orthogonal projectors onto suitable polynomial spaces, which allows the explicit integration of the terms that appear in the bilinear and trilinear forms involved in the scheme for the fluid equations. On the other hand, in order to manipulate the bilinear form associated to the heat equations, we define a suitable projector onto a space of polynomials to deal with the fact that the diffusion tensor, which represents the thermal conductivity, is variable. Next, the corresponding solvability analysis is performed using again appropriate fixed-point arguments. Further, Strang-type estimates are applied to derive the *a priori* error estimates for the components of the virtual element solution as well as for the fully computable projections of them and the postprocessed pressure. The corresponding rates of convergence are also established. Finally, several numerical examples illustrating the performance of the mixed-VEM scheme and confirming these theoretical rates are presented.

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*Key words:* Boussinesq problem, Pseudostress-based formulation, Augmented formulation, Mixed virtual element method, High-order approximations.

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1) Corresponding author

## 1. Introduction

In [36] we developed a mixed-VEM for a pseudostress-velocity formulation of the two-dimensional Navier-Stokes equations. There, we employed a dual-mixed approach based on the introduction of a nonlinear pseudostress linking the usual linear one for the Stokes equations and the convective term. In this way, the resulting continuous scheme is augmented with Galerkin type terms arising from the constitutive and equilibrium equations, and the Dirichlet boundary condition, all them multiplied by suitable stabilization parameters, so that the Banach fixed-point and Lax-Milgram theorems are applied to establish the well-posedness of the continuous scheme (cf. [24]). Regarding the discrete problem we proposed there the simultaneous use of virtual element subspaces for $\mathbf{H}^1$ and $\mathbb{H}(\mathbf{div})$ in order to approximate the velocity and the pseudostress, respectively. Then, the discrete bilinear and trilinear forms involved, their main properties, and the associated mixed virtual scheme were defined, and the corresponding solvability was performed by applying similar techniques to those for the continuous formulation. Other contributions dealing with VEM for nonlinear models include [13, 14, 21, 25, 37, 45]. In particular, a virtual element method employing a low-order approximation of the displacement field is introduced in [13] for nonlinear elastic and inelastic materials. Additionally, a mixed-VEM for quasi-Newtonian Stokes flows is proposed in [21], whereas its extension to a nonlinear Brinkman model of porous media flow is developed in [37]. In turn, a virtual element method dealing with quasilinear elliptic problems is studied in [25]. Finally, an $\mathrm{H}^1$-conforming VEM for the Navier-Stokes equations was introduced in [14], and a nonconforming one was provided in [45].

On the other hand, concerning general numerical methods for approximating the solution of the Boussinesq system, and among the large amount of contributions in the literature, we highlight here [2, 15, 19, 27, 41, 42], which include the primal velocity-pressure-temperature formulations introduced in [2, 15, 19]. In particular, it is established in [15] that the use of any pair of stable Stokes elements for the fluid unknowns and Lagrange elements for the temperature leads to a convergent scheme. In turn, piecewise linear conforming velocities and temperatures, and piecewise constant pressures are proposed in [2], in such a way that the pressure stabilization is carried out by penalizing inter-element jumps. Furthermore, the development of new mixed finite element methods for the Boussinesq model has constituted a very active research topic in recent years [3–5, 29–31]. In particular, an augmented mixed-primal formulation is introduced and analyzed in [29], where the sought quantities are the pseudostress, the velocity, the temperature, and the normal heat flux through the boundary. Under sufficiently small data, it is proved there that when Raviart-Thomas, Lagrange, and discontinuous piecewise finite elements are used to approximate the above unknowns, then the resulting Galerkin method is well-posed and optimally-convergent. Similarly, two formulations for this model, based on a dual-mixed formulation for the momentum equation, and either a primal or a mixed-primal one for the energy equation, are proposed in [30]. In this case, the velocity, the trace-free gradient, and the normal heat flux are approximated by discontinuous piecewise polynomials, whereas Raviart-Thomas and Lagrange elements are employed for the stress and the temperature, which guarantees the stability and the optimal convergence of the finite element methods. In turn, the Boussinesq problem with temperature-dependent parameters was studied in [3] for the two-dimensional case. There, the authors propose an augmented mixed-primal finite element method that approximates the pseudostress tensor with Raviart-Thomas elements of order $k+1$, the velocity and the temperature with Lagrange elements of order $k$, and the vor-