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CONVERGENCE OF NUMERICAL SCHEMES FOR A CONSERVATION EQUATION WITH CONVECTION AND DEGENERATE DIFFUSION*

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Abstract

The approximation of problems with linear convection and degenerate nonlinear diffusion, which arise in the framework of the transport of energy in porous media with thermodynamic transitions, is done using a θ-scheme based on the centred gradient discretisation method. The convergence of the numerical scheme is proved, although the test functions which can be chosen are restricted by the weak regularity hypotheses on the convection field, owing to the application of a discrete Gronwall lemma and a general result for the time translate in the gradient discretisation setting. Some numerical examples, using both the Control Volume Finite Element method and the Vertex Approximate Gradient scheme, show the role of θ for stabilising the scheme.

Mathematics subject classification: 65N30, 35K65.
Key words: Linear convection, Degenerate diffusion, Gradient discretisation method, θ-scheme.

1. Introduction

The development of geothermal energy leads to increasing needs for simulating the displacement of the water in a porous medium, accounting for the liquid-vapour change of phase [5]. This is achieved by writing the system of the conservation equation of the mass of water and that of the conservation of energy, together with a system of equations and inequalities expressing the thermodynamic equilibrium between the two phases when they are simultaneously present [10]. Let us consider a simplification of this system, which may be considered as a

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reasonable approximation in some physical cases:

$$\partial_t(\rho_l(1-S)+\rho_v S) - \operatorname{div}\left(\frac{K}{\mu}(\rho_l(1-S)+\rho_v S)\nabla P\right) = w, \tag{1.1}$$

$$\partial_t(e_l(1-S)+e_v S) - \operatorname{div}\left(\frac{K}{\mu}(e_l(1-S)+e_v S)\nabla P + \Lambda\nabla T\right) = f, \tag{1.2}$$

$$(T < T_e \text{ and } S=0) \text{ or } (T = T_e \text{ and } 0 \le S \le 1) \text{ or } (T > T_e \text{ and } S = 1). \tag{1.3}$$

In the preceding system, the indices l, v respectively stand for the liquid and vapour phases, S is the saturation of the vapour phase (hence $1-S$ is that of the liquid phase), P the pressure assumed to be common for the two fluids (we neglect the capillary pressure), and for $\alpha = l, v$, ρ_α and e_α are respectively the density and the internal energy per mass unit of the phase α, assumed to be given functions of T. In System (1.2)-(1.3), the mobilities of the phases l and v are assumed to be equal to $(1-S)/\mu$ and S/μ, assuming the same viscosity μ for the two phases, and K is the absolute permeability field. The thermal conductivity is denoted by Λ. The right hand sides w and f are respectively the source terms of water and energy. The thermodynamic equilibrium between the two fluid phases l and v is assumed to hold when the temperature is equal to the equilibrium temperature T_e, assumed to be a constant; otherwise, one of the two fluid phases is missing.

Now denoting by $\bar{u} = e_l(1-S)+e_v S$, we notice that, from (1.3), it is possible to express $T - T_e$ as a function ν of $\bar{u}$. For example, if $e_l = C_l(T-T_e)$ and $e_v = L + C_v(T-T_e)$, where L is the latent heat and C_α the thermal capacity of phase α, then there holds,

$$\nu(\bar{u}) = \begin{cases} \dfrac{\bar{u}}{C_l}, & \bar{u} < 0, \\ 0, & 0 \le \bar{u} \le L, \\ \dfrac{\bar{u}-L}{C_v}, & \bar{u} > L. \end{cases}$$

Therefore, denoting by $\boldsymbol{\vec{v}} = -\frac{K}{\mu}\nabla P$ and only focusing on the energy conservation (we assume that the water conservation equation (1.1) is in some way decoupled from this problem), we consider the following linear convection – degenerate diffusion problem, issued from (1.2)-(1.3):

$$\begin{aligned} &\partial_t \bar{u}(\boldsymbol{x},t) \ + \ \operatorname{div}(\, \bar{u}(\boldsymbol{x},t)\boldsymbol{\vec{v}}(\boldsymbol{x},t) - \Lambda(\boldsymbol{x})\nabla\nu(\bar{u}(\boldsymbol{x},t))\,) \\ =&f(\boldsymbol{x},t), \qquad \text{for a.e. } (\boldsymbol{x},t) \in \Omega\times(0,T), \end{aligned} \tag{1.4a}$$

with the initial condition :

$$\bar{u}(\boldsymbol{x},0) = u_{\text{ini}}(\boldsymbol{x}), \qquad \text{for a.e. } \boldsymbol{x} \in \Omega, \tag{1.4b}$$

together with the homogeneous Dirichlet boundary condition :

$$\nu(\bar{u}(\boldsymbol{x},t)) = 0 \quad \text{ on } \partial\Omega\times(0,T). \tag{1.4c}$$