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# MONOLITHIC MULTIGRID FOR REDUCED MAGNETOHYDRODYNAMIC EQUATIONS*

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**Abstract**

In this paper, the monolithic multigrid method is investigated for reduced magnetohydrodynamic equations. We propose a diagonal Braess-Sarazin smoother for the finite element discrete system and prove the uniform convergence of the MMG method with respect to mesh sizes. A multigrid-preconditioned FGMRES method is proposed to solve the magnetohydrodynamic equations. It turns out to be robust for relatively large physical parameters. By extensive numerical experiments, we demonstrate the optimality of the monolithic multigrid method with respect to the number of degrees of freedom.

*Mathematics subject classification:* 65M60, 76W05.
*Key words:* Monolithic multigrid, Magnetohydrodynamic equations, Diagonal Braess-Sarazin smoother, Finite element method.

## 1. Introduction

The incompressible magnetohydrodynamic (MHD) equations governs the dynamics of a charged fluid in the presence of electromagnetic fields. It has broad applications in technology and engineering, such as aluminum electrolysis, electromagnetic pumping, stirring of liquid metals, and flow-quantity measurements based on magnetic induction [1, 9, 12]. The governing model is a coupled system of Navier-Stokes equations and Maxwell's equations. When the magnetic field tends to be saturated or the electric conductivity is relatively small, the model is usually simplified to the reduced MHD (RMHD) equations [19, 22, 27]. In dimensionless form, the stationary RMHD model is given by

$$-\frac{1}{R_e}\triangle \boldsymbol{u} + \nabla p + N(\nabla\phi - \boldsymbol{u}\times\boldsymbol{B})\times\boldsymbol{B} = \boldsymbol{f} \qquad \text{in } \Omega, \tag{1.1a}$$

$$\nabla\cdot\boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{1.1b}$$

$$-\triangle\phi + \nabla\cdot(\boldsymbol{u}\times\boldsymbol{B}) = \chi \qquad \text{in } \Omega, \tag{1.1c}$$

$$\boldsymbol{u} = 0, \quad \phi = \xi \qquad \text{on } \Gamma, \tag{1.1d}$$

where $\Omega\subset\mathbb{R}^3$ is a bounded domain with Lipschitz-continuous boundary $\Gamma := \partial\Omega$. The unknowns are the velocity of fluid $\boldsymbol{u}$, the hydrodynamic pressure $p$, and the electric potential $\phi$. The right-hand sides $\boldsymbol{f}\in\boldsymbol{L}^2(\Omega)$, $\chi\in L^2(\Omega)$ and the applied magnetic field $\boldsymbol{B}\in\boldsymbol{H}^1(\Omega)$ are assumed to be given. The non-dimensional parameters $R_e$ and $N$ are the Reynolds number

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and coupling parameter. The boundary value for $\phi$ satisfies $\xi \in H^{1/2}(\Gamma)$. Here the momentum equation (1.1a) does not contain the convection term $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$. The model can be interpreted as the coupling between the Stokes equations and the electric potential Poisson equation or as the Stokes linearization of time-dependent Navier-Stokes equations where the convective term is treated explicitly in time (see, e.g., [16, 20, 24]).

There are many papers in the literature on numerical solutions of RMHD equations. In [27], Peterson proved the existence and uniqueness of weak solutions of RMHD model and studied its finite element approximation. Layton et al [19] proved $L^2$-error estimates and proposed a two-level method to deal with the nonlinearity. Ervin et al studied a posteriori error estimates for both standard finite element method as well as a two-level Newton finite element method [11]. For time-dependent problems, Yuksel and Ingram gave a comprehensive error analysis for both semi-discrete and fully discrete approximate [35, 36]. In 2007, Ni et al developed consistent and charge-conservative schemes for inductionless MHD equations on both structured and unstructured meshes [25,26]. For theoretical and numerical studies of other MHD models, we refer to [14, 17, 18, 22, 30, 32] and references therein.

After linearization and discretization, the MHD equations usually result in a large and indefinite linear system which is very hard to solve. The study for robust and efficient preconditioners is an important research topic. Among various existing solvers for these types of systems, one may distinguish between block preconditioners and monolithic multigrid (MMG). Block preconditioners exploit inherent block structure of the fully coupled system and utilize existing Poisson solvers as building blocks. The key ingredient is to construct approximate Schur complements (see, e.g., [21, 23, 24, 28, 29]).

The MMG method solves the fully-coupled system not only on the finest level of finite element meshes, but also on coarse levels. Over the past three decades, great success has been achieved in MMG for large-scale multi-physics problems. We refer to [7, 38] for MMG methods for Stokes equations and Navier-Stokes equations and to [2] for MMG method for resistive MHD equations. In [33, 34], Salah et al proposed a fully-coupled multilevel preconditioner for incompressible resistive MHD in the context of fully-implicit time integration and direct-to-steady-state solution. Recently, Adler et al proposed an MMG-preconditioned GMRES method for vector-potential formulation of two-dimensional resistive MHD equations [3]. However, to the authors' knowledge, there are few papers in the literature on the convergence of multigrid (MG) method for MHD equations.

The objective of this paper is to investigate the MMG method for solving the discrete problem of (1.1). Inspired by Braess and Sarazin, we propose a diagonal Braess-Sarazin (DBS) smoother for solving the discrete system. Different from classical Braess-Sarazin smoother for Stokes equations [7, 38], our smoother uses damped Jacobi relaxation for each variable. It is easy to implement and economic in practical computations. A rigorous convergence analysis is presented for the MMG method by verifying its smoothing and approximation properties. Furthermore, we also propose a MMG-preconditioned FGMRES method for the discrete problem. Numerical experiments show that the method is robust for large physical parameters and quasi-optimal with respect to mesh sizes.

The paper is organized as follows. In Section 2, we introduce the variational formulation and present mixed finite element approximation to the RMHD equations. Some preliminary estimates are also presented. In Section 3, we present the MMG algorithm with DBS smoother. In Section 4, we prove the convergence of the MMG method by virtue of smoothing property and approximation property. In Section 5, we present some numerical experiments to demonstrate