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ERROR ESTIMATES FOR SPARSE OPTIMAL CONTROL PROBLEMS BY PIECEWISE LINEAR FINITE ELEMENT APPROXIMATION*

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Abstract

Optimization problems with L^1-control cost functional subject to an elliptic partial differential equation (PDE) are considered. However, different from the finite dimensional l^1-regularization optimization, the resulting discretized L^1-norm does not have a decoupled form when the standard piecewise linear finite element is employed to discretize the continuous problem. A common approach to overcome this difficulty is employing a nodal quadrature formula to approximately discretize the L^1-norm. In this paper, a new discretized scheme for the L^1-norm is presented. Compared to the new discretized scheme for L^1-norm with the nodal quadrature formula, the advantages of our new discretized scheme can be demonstrated in terms of the order of approximation. Moreover, finite element error estimates results for the primal problem with the new discretized scheme for the L^1-norm are provided, which confirms that this approximation scheme will not change the order of error estimates. To solve the new discretized problem, a symmetric Gauss-Seidel based majorized accelerated block coordinate descent(sGS-mABCD) method is introduced to solve it via its dual. The proposed sGS-mABCD algorithm is illustrated at two numerical examples. Numerical results not only confirm the finite element error estimates, but also show that our proposed algorithm is efficient.

Mathematics subject classification: 49N05, 65N30, 49M25, 68W15.
Key words: Finite element method, ABCD method, Approximate discretization, Error estimates.

1. Introduction

In this paper, we study the following linear-quadratic elliptic PDE-constrained optimal control problem with L^1-control cost and piecewise box constraints on the control:

$$\left\{\begin{array}{lll}\min\limits_{(y,u)\in Y\times U_{ad}} & J(y,u)=\frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)}+\frac{\alpha}{2}\|u\|^2_{L^2(\Omega)}+\beta\|u\|_{L^1(\Omega)} & \\ \text{s.t.} & Ly=u+y_r \quad \text{in } \Omega, & \\ & y=0 \qquad\quad \text{on } \partial\Omega, & \end{array}\right. \tag{P}$$

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where $Y := H_0^1(\Omega)$, $U := L^2(\Omega)$, $U_{ad} = \{v(x)|a \leq v(x) \leq b, \text{ a.e. on } \Omega\} \subseteq U$, $\Omega \subseteq \mathbb{R}^n$ ($n = 2$ or 3) is a convex, open and bounded domain with $C^{1,1}$- or polygonal boundary Γ, y_d, $y_r \in C^0(\Omega) \cap H^1(\Omega)$ and the parameters $a \leq 0 \leq b$ and α, $\beta > 0$. Moreover the operator L is a second-order linear elliptic differential operator. It is well-known that L^1-norm could lead to sparse optimal control, i.e. the optimal control with small support. Such an optimal control problem (P) plays an important role for the placement of control devices [1]. In some cases, it is difficult or undesirable to place control devices all over the control domain and one hopes to localize controllers in small and effective regions, the L^1-solution gives information about the optimal location of the control devices.

Let us comment on known results on a-priori analysis of control constrained sparse optimal control problems. For the study of optimal control problems with sparsity promoting terms, as far as we know, the first paper devoted to this study is published by Stadler [1], in which structural properties of the control variables were analyzed in the case of the linear-quadratic elliptic optimal control problem. In 2011, a priori and a posteriori error estimates were first given by Wachsmuth and Wachsmuth in [2] for piecewise linear control discretizations, in which they prove the following result

$$\|u^* - u_h^*\|_{L^2(\Omega)} \leq C(\alpha^{-1}h + \alpha^{-3/2}h^2). \tag{1.1}$$

However, from an algorithmic point of view, the resulting discrete L^1-norm:

$$\|u_h\|_{L^1(\Omega_h)} := \int_{\Omega_h} \Big|\sum_{i=1}^{n} u_i\phi_i(x)\Big|\mathrm{d}x,$$

does not have a decoupled form with respect to the coefficients $\{u_i\}$, where $\phi_i(x)$ are the piecewise linear nodal basis functions which lead to its subgradient $\nu_h \in \partial\|u_h\|_{L^1(\Omega_h)}$ will not belong to a finite-dimensional subspace. Thus, directly solving the corresponding discrete problem will causes many difficulties in numerical calculation. Hence, the authors introduced an alternative discretization of the L^1-norm which relies on a nodal quadrature formula:

$$\|u_h\|_{L_h^1(\Omega_h)} := \sum_{i=1}^{n} |u_i| \int_{\Omega_h} \phi_i(x)\mathrm{d}x. \tag{1.2}$$

About the approximate L^1-norm, based on the error estimates of the nodal interpolation operator, it is easy to show that

$$0 \leq \|u_h\|_{L_h^1(\Omega_h)} - \|u_h\|_{L^1(\Omega_h)} = O(h) \tag{1.3}$$

Obviously, this quadrature incurs an additional error. However, the authors [2] proved that this approximation does not change the order of error estimates.

In a sequence of papers [3, 4], for the non-convex case governed by a semilinear elliptic equation, Casas et al. proved second-order necessary and sufficient optimality conditions. Using the second-order sufficient optimality conditions, the authors provide error estimates of order h w.r.t. the L^∞ norm for three different choices of the control discretization (including the piecewise constant, piecewise linear control discretization and the variational control discretization). It should be pointed that, for the piecewise linear control discretization case, a similar approximation technique to the one introduced by Wachsmuth and Wachsmuth is also used for the discretizations of the L^2 norm and L^1 norm of the control.