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SUB-OPTIMAL CONVERGENCE OF DISCONTINUOUS GALERKIN METHODS WITH CENTRAL FLUXES FOR LINEAR HYPERBOLIC EQUATIONS WITH EVEN DEGREE POLYNOMIAL APPROXIMATIONS*

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Abstract

In this paper, we theoretically and numerically verify that the discontinuous Galerkin (DG) methods with central fluxes for linear hyperbolic equations on non-uniform meshes have sub-optimal convergence properties when measured in the L^2-norm for even degree polynomial approximations. On uniform meshes, the optimal error estimates are provided for arbitrary number of cells in one and multi-dimensions, improving previous results. The theoretical findings are found to be sharp and consistent with numerical results.

Mathematics subject classification: 65M60, 65M15
Key words: Discontinuous Galerkin method, Central flux, Sub-optimal convergence rates

1. Introduction

A fundamental form of energy transmission is wave propagation, which arises in many fields of science, engineering and industry, such as petroleum engineering, geoscience, telecommunication, and the defense industry (see [8, 12]). It is important for these applications to study efficient and accurate numerical methods to solve wave propagation problems. Experience reveals that energy-conserving numerical methods, which conserve the discrete approximation of energy, are favorable, because they are able to maintain the phase and shape of the waves more accurately, especially for long-time simulation.

Various numerical approximations of wave problems modeled by linear hyperbolic systems can be found in the literature. Here, we will focus on the classical Runge-Kutta DG method of Cockburn and Shu [6]. There are several approaches to obtain an optimal, energy conserving DG method. Chung and Engquist [4] presented an optimal, energy conserving DG method for the acoustic wave equation on staggered grids. Chou et al. [3] proposed an optimal energy conserving DG using alternating fluxes for the second order wave equation. More recently, Fu

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and Shu [9] developed an optimal energy conserving DG method by introducing an auxiliary zero function.

As is well known, the simplest energy conserving DG method for hyperbolic equations is the one using central fluxes. However, it has sub-optimal convergence of order k measured in the L^2-norm when piece-wise polynomials of an *odd* degree k are used; see, e.g. [15]. When k is even, we usually observe higher convergence rates than kth order for a general regular non-uniform meshes, such as random perturbation over an uniform mesh, see section 4. In fact, many papers have mentioned that the optimal convergence rates can be observed when even degree polynomials are used; see for example [1, 2, 7, 15]. In this paper, we provide a counter example to show that the scheme only has sub-optimal error accuracy of order k for a regular non-uniform mesh, when k is even. We refer to the work of Guzmán and Rivière [11] in which they constructed a special mesh sequence to produce the sup-optimal accuracy for the non-symmetric DG methods for elliptic problems when k is odd. For uniform meshes, the classical DG scheme with the central flux does have the optimal convergence rate $k+1$, observed in the numerical experiments and proved theoretically under the condition that the number of cells in the mesh is *odd* [1, 15]. In this paper, we provide a new proof which is available for arbitrary number of cells and dimensions for linear hyperbolic equations. We have used the shifting technique [13, 14] to construct the special local projection to obtain the optimal error estimate on uniform meshes. We also numerically find the superconvergence phenomenon for the cell averages and numerical fluxes.

The outline of the paper is as follows. In section 2, we review the DG scheme for hyperbolic equations with central fluxes and give the error estimates for the semi-discrete version in one dimension. We extend our analysis to multi-dimensions in section 3. In section 4, we give numerical examples to show the sub-optimal convergence for non-uniform meshes and optimal convergence for uniform meshes in both one and two-dimensional cases. Finally, we give concluding remarks in section 5. Some of the technical proof of the lemmas and propositions is included in the Appendix A.

2. One Dimensional Problems

We consider the following one dimensional linear hyperbolic equation

$$\begin{cases} u_t + u_x = 0, & x \in [0,1],\ t \geq 0 \\ u(x,0) = u_0(x), & x \in [0,1], \end{cases} \tag{2.1}$$

with periodic boundary condition. We first introduce the usual notations of the DG method. For a given interval $\Omega = [0,1]$ and the index set $\mathbb{Z}_N = \{1, 2, \ldots, N\}$, the usual DG mesh $\mathcal{I}_N$ is defined as:

$$0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \ldots < x_{N+\frac{1}{2}} = 1. \tag{2.2}$$

We denote

$$I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}), \quad x_j = \frac{1}{2}(x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}}), \quad h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}, \quad j \in \mathbb{Z}_N. \tag{2.3}$$

We also assume the mesh is regular, i.e., the ratio between the maximum and minimum mesh sizes shall stay bounded during mesh refinements. That means there exists a positive constant $\sigma \geq 1$, such that,