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IMPLICIT-EXPLICIT RUNGE-KUTTA-ROSENBROCK METHODS WITH ERROR ANALYSIS FOR NONLINEAR STIFF DIFFERENTIAL EQUATIONS*

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Abstract

Implicit-explicit Runge-Kutta-Rosenbrock methods are proposed to solve nonlinear stiff ordinary differential equations by combining linearly implicit Rosenbrock methods with explicit Runge-Kutta methods. First, the general order conditions up to order 3 are obtained. Then, for the nonlinear stiff initial-value problems satisfying the one-sided Lipschitz condition and a class of singularly perturbed initial-value problems, the corresponding errors of the implicit-explicit methods are analysed. At last, some numerical examples are given to verify the validity of the obtained theoretical results and the effectiveness of the methods.

Mathematics subject classification: 65L04, 65L20, 65L06.
Key words: Stiff differential equations, Implicit-explicit Runge-Kutta-Rosenbrock method, Order conditions, Convergence.

1. Introduction

The systems of stiff ordinary differential equations (ODEs) are important mathematical models. They often appear in many fields of science and engineering and come also from space discretization of some initial-boundary value problems of partial differential equations (see, e.g., [1–6, 10–15, 21–23, 28–31]. In particular, many stiff ODEs can be rewritten in additive form whose terms have different stiffness properties, for example, their functions on the right side can be split to stiff part and nonstiff part.

For solving numerically stiff ODEs, Runge-Kutta (RK) methods are a common choice. On the one hand, explicit RK methods require less computing efforts and can be implemented easily, but the demand for their stability leads to the strict time step constraint. On the other hand, many implicit RK methods are often of good stability property, but need to solve the systems of nonlinear algebraic equations.

A good compromise is to apply linearly implicit Rosenbrock-type RK schemes, so-called W-methods (see, e.g., [2, 12, 13, 15, 16, 19, 24, 25, 27, 32]), which can avoid the solution of the systems of nonlinear algebraic equations and only require the solution of linear algebraic systems at each time step. Bassi et al. [2] focused on the implementation and assessment of linearly

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implicit Rosenbrock schemes as time integrators for the high-order discontinuous Galerkin space discretization of the compressible and incompressible Navier-Stokes equations. González-Pinto et al. [12] considered a new class of AMF-type W-methods for the time integration of a large system of ODEs and linear parabolic problems with mixed derivatives and constant coefficients.

For combining the advantages of explicit and implicit RK methods, implicit-explicit (IMEX) RK methods have been proposed and become a very active area of research (see, e.g., [3-11, 17, 18, 20–22, 30, 31, 34–37]). For instance, Ascher et al. [1] constructed the efficient IMEX RK methods with better stability regions than the best known IMEX multistep schemes over a wide parameter range. Boscarino et al. [3–7] focused on IMEX RK methods for differential-algebraic systems, hyperbolic systems, kinetic equations, fully nonlinear relaxation problems and stiff problems. Izzo et al. [20] investigated the construction of highly stable IMEX RK methods up to the order $p = 4$, where the implicit schemes are A-stable and the explicit schemes has strong stability property. Yue et al. [35] were devoted to the nonlinear stability and B-convergence of additive RK (ARK) methods for nonlinear stiff problems with multiple stiffness. Other efficient IMEX RK methods and their applicitions can be found in [11, 31] etc.

For reducing further computing efforts of IMEX RK methods, linearized implicit RK methods (such as Rosenbrock methods) can replace the implicit RK methods in IMEX RK methods. Ullrich et al. [28] introduced an operator-split Runge-Kutta-Rosenbrock (RKR) time discretization strategy for nonhydrostatic atmospheric models, and its temporal accuracy up to order 3 is achieved. Higueras et al. [18] considered additive semi-implicit RK (ASIRK) methods for stiff additive differential systems by combining diagonally implicit RK methods or linearized implicit RK methods with explicit RK methods, and constructed two 2-order 3-stage ASIRK schemes with low-storage requirements. Other efficient IMEX linearized RK methods can refer to [37]. But these works only considered the construction of efficient algorithms and the analysis of the linear stability, and no rigorous global error analysis was made.

Motivated by the above discussion, we consider further rigorous global error analysis and the construction of efficient algorithms for IMEX RKR methods for nonlinear stiff ODEs. The order conditions up to order 3 are obtained. For the nonlinear stiff initial-value problems satisfying the one-sided Lipschitz condition and a class of singularly perturbed initial-value problems, the errors of these methods are analysed rigorously. Moreover, 2-order 3-stage IMEX RKR methods are proposed and shown to be efficient, and they can overcome the severe time step restriction of explicit schemes and require just one Jacobian matrix evaluation per time step, thus the overall computational efforts are reduced obviously.

The rest of the paper is organized as follows. In Section 2, we present the IMEX RKR methods by combining implicit Rosenbrock methods with explicit RK methods for stiff ODEs. The order conditions of the IMEX RKR methods are given in Section 3. In Section 4, we give the convergence results of the IMEX RKR methods for the nonlinear stiff initial-value problems of ODEs with one-sided Lipschitz condition. In Section 5, the convergence results of the IMEX RKR methods for a class of singularly perturbed initial-value problems are obtained. In Section 6, some numerical examples are given to verify the obtained theoretical results.

2. IMEX RKR Methods

Consider the initial-value problems of nonlinear stiff ODEs

$$\begin{cases} y'(t) = F(t, y(t)) = f(t, y(t)) + g(t, y(t)), & t \in [0, T_e], \\ y(0) = y_0, & y_0 \in R^m, \end{cases} \tag{2.1}$$