Journal of Computational Mathematics
Vol.29, No.4, 2021, 556–573.
http://www.global-sci.org/jcm
doi:10.4208/jcm.2003-m2019-0226

CHARACTERISATION OF RATIONAL AND NURBS DEVELOPABLE SURFACES IN COMPUTER AIDED DESIGN*

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Abstract

In this paper we provide a characterisation of rational developable surfaces in terms of the blossoms of the bounding curves and three rational functions Λ, M, ν. Properties of developable surfaces are revised in this framework. In particular, a closed algebraic formula for the edge of regression of the surface is obtained in terms of the functions Λ, M, ν, which are closely related to the ones that appear in the standard decomposition of the derivative of the parametrisation of one of the bounding curves in terms of the director vector of the rulings and its derivative. It is also shown that all rational developable surfaces can be described as the set of developable surfaces which can be constructed with a constant Λ, M, ν . The results are readily extended to rational spline developable surfaces.

Mathematics subject classification: 65D17, 68U07.
Key words: NURBS, Bézier, Rational, Spline, NURBS, Developable surfaces.

1. Introduction

Ruled surfaces are useful since the simplest way to interpolate a surface patch between two given curves is to link them with straight segments. Ruled surfaces have non-positive Gaussian curvature, since in general the straight lines that they contain are not lines of curvature of the surface. In developable surfaces the straight lines are one of the families of lines of curvature and hence these surfaces have null Gaussian curvature.

Mathematically, this means that developable surfaces are isometric to the plane. Extrinsic, but not intrinsic, curvature arises from the way these surfaces are embedded in space. Since distances, areas and angles are conserved on embedding the surfaces in space, this means that developable surfaces are plane patches which have been folded or cut, but not deformed in any other fashion. Rolling pieces of planes in cones and cylinders are the most obvious ways of achieving this, but there are more general and less intuitive ways.

For such reason developable surfaces are valuable for applications in industry. Developable surfaces model the way the pages of a book are folded [1], the forms of facades in architecture [2] or the shapes adopted by garments [3] with plane patterns. They are also useful in industries related to building with sheets of steel or wood, such as naval industry [4–6], or even automobile industry [7]. In the case of steel this means that parts of the hull of a ship can be modeled with developable surfaces and can be produced by folding machines without application of heat, reducing costs and modifications of the metallic structure.

Since Gaussian curvature is the quotient of the determinants of the fundamental forms of the surface, its calculation involves non-linear combinations of the derivatives of the parametrisation of the surface. If we think of applications to Computer Aided Geometric Design, this

* Received October 14, 2019 / Revised version received January 19, 2020 / Accepted March 2, 2020 / Published online May 25, 2021 /

translates into non-linear expressions in terms of the control points and weights of the surface. An extensive review on this issue appears in [8].

In the case of rational surfaces, conditions for null Gaussian curvature can be solved for low degrees [9], but there are other approaches to this issue. For instance, restriction to boundary curves on parallel planes simplifies the problem [10, 11].

A geometrically appealing approach relies on projective geometry. In dual space points are planes in space. Since developable surfaces can be viewed as envelopes of one-parametric families of planes, dual space appears as a natural framework [12–14], though the actual control points lie on ordinary space. In [15] the null Gaussian curvature condition is written in terms of quadratic equations in order to devise a constraint useful for interactive modeling.

Also within the NURBS framework, the properties of the de Casteljau algorithm have been explored for constructing developable surfaces [16]. In [17, 18] Bézier developable patches are constructed by applying affine transformations to the first cell of the control net of the patch. It is shown in [19] that this construction produces all Bézier developable surfaces with a polynomial edge of regression. This construction has been extended to spline developable surfaces [20, 21] and to Bézier triangular surfaces [22].

Another interesting approach for designing approximately developable surfaces ia based on the use of the convex hulls of the boundary [3]. Other approximations may be found in [6, 23].

Most recently [24] presents a new approach grounded on the characterisation of developable surfaces as surfaces parametrised by orthogonal sets of geodesics. [25] suggests producing developable triangular meshes in order to design developable surfaces. [26] contructs developable patches bounded by two curves, reparametrising one of the curves.

Since the standard of Computer Aided Design (CAD) is based on the use of rational B-spline curves and surfaces, one would require a description of rational developable surfaces within this framework. That is, involving the elements that are used in design for defining curves and surfaces. such as control points, knots and weights. It would be interesting hence to extend Aumann's approach [17, 19] from polynomial to rational developable surfaces in order to comply with the whole NURBS framework. The main advantage of this approach is the use of the elements which are used in CAD applications.

This paper is organised as follows. Section 2 is devoted to an introduction to developable surfaces as envelopes of families of planes and their classification in terms of their edge of regression. Section 3 provides a characterization of rational developable surfaces based on the de Casteljau algorithm, in terms of three rational functions, Λ, M, σ. Section 4 discusses a useful way of parametrising rational ruled surfaces, which allows an interpretation for Λ, M, σ. Function σ can become trivial by a suitable choice of a global factor for the parametrisation. The main properties of rational developable surfaces in our framework are described in Section 5. The edge of regression of rational developable surfaces is calculated in closed form in Section 6. It is shown that it is a rational curve of degree $n+1$ for rational developable patches with bounding curves of degree n in the case of constant Λ, M, σ. The converse is also true, that is, every rational developable surface admits surface patches with constant Λ, M, σ. This suggests that one may start with constant Λ, M, σ patches and modify the length of the rulings afterwards to adapt them to one's purposes. The construction of constant Λ, M, $\sigma = 1$ patches is derived in Section 7. Examples are shown at the end of the paper.