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STRONG CONVERGENCE OF A FULLY DISCRETE FINITE ELEMENT METHOD FOR A CLASS OF SEMILINEAR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS WITH MULTIPLICATIVE NOISE*

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Abstract

This paper develops and analyzes a fully discrete finite element method for a class of semilinear stochastic partial differential equations (SPDEs) with multiplicative noise. The nonlinearity in the diffusion term of the SPDEs is assumed to be globally Lipschitz and the nonlinearity in the drift term is only assumed to satisfy a one-sided Lipschitz condition. These assumptions are the same ones as the cases where numerical methods for general nonlinear stochastic ordinary differential equations (SODEs) under "minimum assumptions" were studied. As a result, the semilinear SPDEs considered in this paper are a direct generalization of these nonlinear SODEs. There are several difficulties which need to be overcome for this generalization. First, obviously the spatial discretization, which does not appear in the SODE case, adds an extra layer of difficulty. It turns out a spatial discretization must be designed to guarantee certain properties for the numerical scheme and its stiffness matrix. In this paper we use a finite element interpolation technique to discretize the nonlinear drift term. Second, in order to prove the strong convergence of the proposed fully discrete finite element method, stability estimates for higher order moments of the H^1-seminorm of the numerical solution must be established, which are difficult and delicate. A judicious combination of the properties of the drift and diffusion terms and some nontrivial techniques is used in this paper to achieve the goal. Finally, stability estimates for the second and higher order moments of the L^2-norm of the numerical solution are also difficult to obtain due to the fact that the mass matrix may not be diagonally dominant. This is done by utilizing the interpolation theory and the higher moment estimates for the H^1-seminorm of the numerical solution. After overcoming these difficulties, it is proved that the proposed fully discrete finite element method is convergent in strong norms with nearly optimal rates of convergence. Numerical experiment results are also presented to validate the theoretical results and to demonstrate the efficiency of the proposed numerical method.

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1. Introduction

We consider the following initial-boundary value problem for general semilinear stochastic partial differential equations (SPDEs) with function-type multiplicative noise:

$$du = \big[\Delta u + f(u)\big]\,dt + g(u)\,dW(t), \qquad \text{in } \mathcal{D}\times(0,T), \tag{1.1}$$

$$\frac{\partial u}{\partial \nu} = 0, \qquad \text{on } \partial\mathcal{D}\times(0,T), \tag{1.2}$$

$$u(\cdot,0) = u_0(\cdot), \qquad \text{in } \mathcal{D}. \tag{1.3}$$

Here $\mathcal{D}\subset\mathbf{R}^d$ $(d=1,2,3)$ is an open bounded domain with smooth boundary, $W:\Omega\times(0,T)\to\mathbf{R}$ denotes the standard Wiener process on the filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t : t\geq 0\},\mathbb{P})$, and $f,g\in C^1$ are two given functions and $f(u)$ takes the form

$$f(u) = c_0u - c_1u^3 - c_2u^5 - c_3u^7 - \cdots, \tag{1.4}$$

where $c_i\geq 0, i=0,1,2,\cdots$. For the sake of clarity, we only consider the case $f(u)=u-u^q$ in this paper, where $q\geq 3$ is an odd integer (it is trivial when $f(u)=c_0u$). We remark that similar results still hold for the general nonlinear function $f(u)$ in (1.4), and when $f(u)=\frac{1}{\epsilon^2}(u-u^3)$, (1.1) is known as the stochastic Allen-Cahn equation with function-type multiplicative noise and interaction length ϵ (see [28]). We also assume that g is globally Lipschitz and satisfies the growth condition, that is, there exist constants $\kappa_1>0$ and $C>0$ such that

$$|g(a)-g(b)|\leq \kappa_1|a-b|, \tag{1.5}$$

$$|g(a)|^2\leq C(1+a^2). \tag{1.6}$$

By (1.6), we get

$$|g(a)\,a|\leq C(1+a^2). \tag{1.7}$$

Under the above assumptions for the drift term and the diffusion term, it can be proved in [15] that there exists a unique strong variational solution u such that

$$\begin{aligned}(u(t),\phi) = (u(0),\phi) &- \int_0^t \Big(\nabla u(s),\nabla\phi\Big)\,ds + \int_0^t \big(f(u(s)),\phi\big)\,ds\\ &+ \int_0^t (g(u),\phi)\,dW(s) \qquad \forall\,\phi\in H^1(\mathcal{D})\end{aligned} \tag{1.8}$$

holds $\mathbb{P}$-almost surely. Moreover, when the initial condition u_0 is sufficiently smooth, the following stability estimate for the strong solution u holds:

$$\sup_{t\in[0,T]}\mathbb{E}\Big[\|u(t)\|_{H^2}^{2q}\Big] + \sup_{t\in[0,T]}\mathbb{E}\Big[\|u(t)\|_{L^{4q-2}}^{4q-2}\Big]\leq C, \tag{1.9}$$

where q is the exponent in the nonlinear term of $f(u)=u-u^q$.

Clearly, when the Δu term in (1.1) is dropped, the PDE reduces to a stochastic ODE. A convergence theory for numerical approximations for this stochastic ODE was established long ago (see [29] and [30]) under the global Lipschitz assumptions on f and g. Later, the convergence was proved in [18] under a weaker condition on f known as a one-sided Lipschitz condition in the sense that there exists a constant $\mu>0$ such that

$$(a-b, f(a)-f(b))\leq \mu(a-b)^2 \qquad \forall a,b\in\mathbb{R}. \tag{1.10}$$