Journal of Computational Mathematics
Vol.39, No.4, 2021, 621–632.
http://www.global-sci.org/jcm
doi:10.4208/jcm.2005-m2019-0298

CONVERGENCE ANALYSIS ON SS-HOPM FOR BEC-LIKE NONLINEAR EIGENVALUE PROBLEMS*

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Abstract

Shifted symmetric higher-order power method (SS-HOPM) has been proved effective in solving the nonlinear eigenvalue problem oriented from the Bose-Einstein Condensation (BEC-like NEP for short) both theoretically and numerically. However, the convergence of the sequence generated by SS-HOPM is based on the assumption that the real eigenpairs of BEC-like NEP are finite. In this paper, we will establish the point-wise convergence via Łojasiewicz inequality by introducing a new related sequence.

Mathematics subject classification: 15A18, 15A69, 90C26.
Key words: Nonlinear eigenvalues, Bose-Einstein Condensation, SS-HOPM, Point-wise convergence, Łojasiewicz inequality.

1. Introduction

Nonlinear eigenvalue problem originated from the Bose-Einstein condensates (BECs) [1–3] is recently well known to be an important and active field [4–8] in quantum physics. Its discrietized form can be described as

$$\begin{cases} \mathcal{A}\mathbf{x}^3 + \mathbf{B}\mathbf{x} = \lambda\mathbf{x} \\ ||\mathbf{x}||_2 = 1, \end{cases} \tag{1.1}$$

where $\mathcal{A} \in \mathbb{R}^{n\times n\times n\times n}$ is a symmetric 4th-order tensor, $\mathbf{B} \in \mathbb{R}^{n\times n}$ is a symmetric matrix, $\mathbf{x} \in \mathbb{R}^n$ is a vector. It can be easily verified that the nonlinear eigenvalue problem (1.1) can be viewed as the the KKT system or first-order necessary condition of the following nonconvex optimization problem

$$\begin{aligned} \min \quad & \frac{1}{2}\mathcal{A}\mathbf{x}^4 + \mathbf{x}^\top\mathbf{B}\mathbf{x} \\ s.t. \quad & ||\mathbf{x}||_2 = 1. \end{aligned} \tag{1.2}$$

Actually, (1.2) is a discrete form of the energy functional form of BECs [9]. From [10], we know that $(\lambda, \mathbf{x})$ is an eigenpair of nonlinear eigenvalue problem (1.1) if and only if $\mathbf{x}$ is a constraint stationary point of (1.2).

* Received December 18, 2019 / Revised version received March 16, 2020 / Accepted May 12, 2020 / Published online May 25, 2021 /
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Hu et al. [9] pointed out that it is a numerical challenge to solve (1.2) efficiently due to the large number of variables and the possible indefiniteness of the Hessian matrix. They have shown that the BEC problem is NP-hard via establishing its relation with the partition problem. So the general BEC-like NEP is NP-hard.

Generally, the approaches for solving BEC-like problem can be divided into numerical methods [11, 12] and optimization methods [7–9, 13]. The shifted symmetric higher-order power method (SS-HOPM) has been proved effective for solving such nonconvex optimization problem in [14]. However the point-wise convergence of SS-HOPM for BEC-like NEP has not been proven yet.

Łojasiewicz inequality [15] has been proven to be an useful tool in analyzing the local convergence of nonconvex optimization problems [16–19]. Based on the Łojasiewicz gradient inequality, Uschmajew has given the point-wise convergence of Gauss-Seidel higher-order power method [20] with results in [21]. The key is to transform the constrained optimization to a proper unconstrained optimization as in [21]. Luo and Yang proved the point-wise convergence of SS-HOPM for high-order tensor eigenpairs with the similar approach [22]. Specifically, they first defined a new sequence based on the sequence generated by SS-HOPM and corresponding analytic function. Then with Łojasiewicz inequality, they showed the global convergence of the new established sequence, which in return ensured the point-wise convergence of the original sequence. Motivated by the work mentioned above, we intend to enhance the convergence result in [14], i.e, to prove the point-wise convergence of SS-HOPM for BEC-like NEP.

The rest of this paper is outlined as follows. Section 2 presents some preliminaries about the basic notation and SS-HOPM. In Section 3, we establish the point-wise convergence of $\{\mathbf{x}_k\}$ generated by SS-HOPM for BEC-like NEP. Conclusions are provided in Section 4.

2. Notation and Preliminaries

Throughout this paper, we exclusively consider the tensor notation introduced in [23]. In particularly, vectors are denoted by boldface lowercase letters, e.g., $\mathbf{a}$. Matrices are denoted by boldface capital letters, e.g., $\mathbf{A}$. Higher-order tensors are denoted by Euler script letters, e.g., $\mathcal{A}$. Scalars are denoted by lowercase letters, e.g., a. Let Σ denote the unit sphere on $\mathbb{R}^n$, i.e., $\Sigma = \{\mathbf{x} \in \mathbb{R}^n : \ \|\mathbf{x}\| = 1\}$.

2.1. Tensors and tensor eigenpair

A *tensor* is a multidimensional array. Its order is the number of its dimensions. An Nth-order tensor is denoted as $\mathcal{A} \in \mathbb{R}^{I_1\times\cdots\times I_N}$, whose $(i_1, \cdots, i_N)$ element is $a_{i_1\cdots i_N}, 1 \leq i_k \leq I_k, k = 1, \cdots, N$. Specifically, for $N = 1$ and $N = 2$, tensors are vectors and matrices respectively.

Definition 2.1 (Symmetric tensor [24]). *A tensor* $\mathcal{A} \in \mathbb{R}^{\overbrace{n\times\cdots\times n}^{m}}$ *is symmetric if*

$$a_{i_{p(1)}\cdots i_{p(m)}} = a_{i_1\cdots i_m} \quad \text{for all } \ i_1\cdots i_m \in \{1, \cdots, n\} \ \text{ and } \ p \in \Pi_m,$$

where Π_m *denotes the set of all permutations of* $(1, \cdots, m)$.

Definition 2.2 (Symmetric tensor-vector multiply). *Let* $\mathcal{A} \in \mathbb{R}^{\overbrace{n\times\cdots\times n}^{m}}$ *be symmetric and* $\mathbf{x} \in \mathbb{R}^n$. *Then for* $0 \leq r \leq m-1$, *the* $(m-r)$*-times product of tensor* $\mathcal{A}$ *with the vector* $\mathbf{x}$