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A FAST COMPACT DIFFERENCE METHOD FOR TWO-DIMENSIONAL NONLINEAR SPACE-FRACTIONAL COMPLEX GINZBURG-LANDAU EQUATIONS*

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Abstract

This paper focuses on a fast and high-order finite difference method for two-dimensional space-fractional complex Ginzburg-Landau equations. We firstly establish a three-level finite difference scheme for the time variable followed by the linearized technique of the nonlinear term. Then the fourth-order compact finite difference method is employed to discretize the spatial variables. Hence the accuracy of the discretization is $\mathcal{O}(\tau^2 + h_1^4 + h_2^4)$ in L_2-norm, where τ is the temporal step-size, both h_1 and h_2 denote spatial mesh sizes in x- and y- directions, respectively. The rigorous theoretical analysis, including the uniqueness, the almost unconditional stability, and the convergence, is studied via the energy argument. Practically, the discretized system holds the block Toeplitz structure. Therefore, the coefficient Toeplitz-like matrix only requires $\mathcal{O}\big(M_1M_2\big)$ memory storage, and the matrix-vector multiplication can be carried out in $\mathcal{O}\big(M_1M_2(\log M_1 + \log M_2)\big)$ computational complexity by the fast Fourier transformation, where M_1 and M_2 denote the numbers of the spatial grids in two different directions. In order to solve the resulting Toeplitz-like system quickly, an efficient preconditioner with the Krylov subspace method is proposed to speed up the iteration rate. Numerical results are given to demonstrate the well performance of the proposed method.

Mathematics subject classification: 26A33, 35R11, 65M06, 65M12.
Key words: Space-fractional Ginzburg-Landau equation, Compact scheme, Boundedness, Convergence, Preconditioner, FFT.

1. Introduction

In this paper, we develop a fast high-order compact finite difference scheme for two-dimensional space-fractional complex Ginzburg-Landau equations [32] in the truncated domain as follows

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$$\partial_t u-(\nu+\mathbf{i}\eta)(\partial_x^\alpha+\partial_y^\beta)u+(\kappa+\mathbf{i}\zeta)|u|^2u-\gamma u=0,\qquad (x,y)\in\Omega,\ 0<t\leqslant T,\tag{1.1}$$
$$u(x,y,t)=0,\qquad (x,y)\in\mathbb{R}^2\backslash\Omega,\ 0<t\leqslant T,\tag{1.2}$$
$$u(x,y,0)=\varphi(x,y),\qquad (x,y)\in\mathbb{R}^2,\tag{1.3}$$

where $1<\alpha,\beta\leqslant 2,\ \nu>0,\ \kappa>0,\ \eta,\ \zeta,\ \gamma$ are given real constants, $\Omega=(x_l,x_r)\times(y_d,y_u)$ is the rectangular region with the boundary $\partial\Omega$, $\mathbf{i}=\sqrt{-1}$ is the imaginary unit, $u(x,y,t)$ is the complex-valued function, and $\varphi(x,y)$ is a given smooth function with compact support vanishing in $\mathbb{R}^2\backslash\Omega$. Furthermore, ∂_x^α in (1.1) denotes the Riesz fractional derivative operator and is defined as [8]

$$\partial_x^\alpha u(x,y,t)=-\frac{1}{2\cos(\alpha\pi/2)\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty}|x-\xi|^{1-\alpha}u(\xi,y,t)d\xi,$$

or, equivalently,

$$\partial_x^\alpha u(x,y,t)=-\frac{1}{2\cos(\alpha\pi/2)}\left[{}_{-\infty}D_x^\alpha u(x,y,t)+{}_xD_{+\infty}^\alpha u(x,y,t)\right],$$

where ${}_{-\infty}D_x^\alpha u(x,y,t)$ denotes the left Riemann-Liouville fractional derivative [25]

$${}_{-\infty}D_x^\alpha u(x,y,t)=\frac{1}{\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_{-\infty}^{x}\frac{u(\xi,y,t)}{(x-\xi)^{\alpha-1}}d\xi,$$

and ${}_xD_{+\infty}^\alpha u(x,y,t)$ denotes the right Riemann-Liouville fractional derivative

$${}_xD_{+\infty}^\alpha u(x,y,t)=\frac{1}{\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_{x}^{+\infty}\frac{u(\xi,y,t)}{(\xi-x)^{\alpha-1}}d\xi.$$

Analogously,

$$\partial_y^\beta u(x,y,t)=-\frac{1}{2\cos(\beta\pi/2)\Gamma(2-\beta)}\frac{\partial^2}{\partial y^2}\int_{-\infty}^{\infty}|y-\xi|^{1-\beta}u(x,\xi,t)d\xi$$

is defined.

Problems (1.1)–(1.3) were firstly proposed by Tarasov and Zaslavsky in recent years [30,31] due to the fast development of fractional quantum mechanics [13,14], which are mainly related to the quantum phenomena in fractal environments. Compared with the conventional complex Ginzburg-Landau equations arising from the fractality of the Brownian trajectories [1], fractional complex Ginzburg-Landau equations originate from the variational Euler-Lagrange equation for fractal media over the Lévy paths [7,35]. In the past few years, the theoretical properties of the fractional complex Ginzburg-Landau equations have been extensively investigated; for example, the well-posedness, dynamics and nviscid limit behavior of solution [9,10,26], the asymptotic dynamics for two-dimensional case and the random attractor for the multiplicative noise [19–21], the asymptotic analysis in the bounded domains [23], and exact and soliton solutions [2,18,28].

There are a lot of works on the numerical solutions of the fractional complex Ginzburg-Landau equations in the literature, specially for the one-dimensional cases; see [11,15,16,33,34,38]. Nevertheless, only a few investigations for the multi-dimensional cases with higher-order discretizations. Mohebbi [24] proposed a numerical algorithm based on Fourier spectral