Journal of Computational Mathematics
Vol.39, No.5, 2021, 727–737.

http://www.global-sci.org/jcm
doi:10.4208/jcm.2002-m2019-0213

NUMERICAL ANALYSIS OF CRANK-NICOLSON SCHEME FOR THE ALLEN-CAHN EQUATION*

Qianqian Chu
Department of Mathematics, Yanbian University, Yanji 133002, China
Department of Mathematics Science, Harbin Engineering University, Harbin 150001, China
Email: 1214181195@qq.com
Guanghui Jin
Department of Mathematics, Yanbian University, Yanji 133002, China
Email: jinguanghui@ybu.edu.cn
Jihong Shen
Department of Mathematics Science, Harbin Engineering University, Harbin 150001, China
Email: shenjihong@hrbeu.edu.cn
Yuanfeng Jin[1)]
Department of Mathematics, Yanbian University, Yanji 133002, China
Email: yfkim@ybu.edu.cn

Abstract

We consider numerical methods to solve the Allen-Cahn equation using the second-order Crank-Nicolson scheme in time and the second-order central difference approach in space. The existence of the finite difference solution is proved with the help of Browder fixed point theorem. The difference scheme is showed to be unconditionally convergent in L_∞ norm by constructing an auxiliary Lipschitz continuous function. Based on this result, it is demonstrated that the difference scheme preserves the maximum principle without any restrictions on spatial step size and temporal step size. The numerical experiments also verify the reliability of the method.

Mathematics subject classification: 65M06, 65M12.
Key words: Allen-Cahn Equation, Crank-Nicolson scheme, Maximum principle, Convergence.

1. Introduction

Phase field model is a mathematical model which is described by partial differential equations. The application of diffuse phase field model on interface movement and the generation of mesoscale morphology has always been the hotspot. To describe the anti-phase boundary motion in the crystal, Allen and Cahn [2] introduced the Allen-Cahn equation. The equation is an important equation describing fluid dynamics and reaction diffusion in material science, the mathematical model was also used in describing many diffusion phenomena, such as the competition, exclusion of biological population, and the migration process of river beds. However, there is no exact solution for such phase field models, so numerical methods have played a particularly important role in various simulations. Many works have been done on the numerical approximation of these phase field models, including the finite difference method, the finite element method and the spectral method.

* Received September 18, 2019 / Revised version received January 19, 2020 / Accepted February 19, 2020 / Published online June 18, 2021 /
1) Corresponding author

This paper is concerned with the numerical approximation of Crank-Nicolson Scheme for the one-dimensional nonlinear Allen-Cahn equation with the initial-boundary value condition which contains small perturbation parameters and strong nonlinearity.

$$u_t = \varepsilon^2 u_{xx} - f(u), \qquad x \in (a,b),\ t \in [0,T], \tag{1.1}$$

$$u(x,0) = u_0(x), \qquad x \in (a,b), \tag{1.2}$$

$$u(a,t) = u(b,t) = 0, \qquad t \in [0,T], \tag{1.3}$$

where the parameter $\varepsilon > 0$ represents the interface width, and the nonlinear term is taken as the polynomial double-well potential $f(u) = u^3 - u$.

In 1992, the exact solution of the Allen Cahn equation that satisfies the maximum principle and energy stability is given by Evance [5]. Zhang and Hou [13] studied finite difference scheme for the one-dimensional Allen-Cahn equation in 2006, using the second-order nonlinear correction Crank-Nicolson format in the time, proved that the difference scheme satisfies the discrete maximum principle and energy stability for the arbitrary stepsize. In 2010, Xu considered the Neumann conditional boundary value problem for one-dimensional and two-dimensional Allen-Cahn equations, advanced semi-implicit fully discrete dissipative finite difference scheme was proposed. Note that the main results in [12] established the implicit-explicit discretization, proved that the numerical solution preserves the maximum principle and the defined energy function is decreasing in time if $0 < \tau \le \frac{1}{2}$. Recently, Hou et al. [7] considered the linear second-order approximation preserving energy stability when $0 \le \tau \le \min\{\frac{1}{4},\ \frac{h^2}{2d\varepsilon^2}\}$. In 2017, Hou and Tang [7] established the second-order Crank-Nicolson scheme in time and second-order central difference approximation in space for fractional-in-space Allen-Cahn equations, the numerical solutions satisfied discrete maximum principle under $0 \le \tau \le \min\{\frac{1}{2},\ \frac{h^\alpha}{2d\varepsilon^2}\}$; Basing on the maximum stability, they investigated the nonlinear energy stability and the corresponding error estimates, proposed a nonlinear iteration algorithm last.

It is known that the solution of the Allen-Cahn equation satisfies the maximum principle. However, there have no rigorous ℓ^∞-stability analogue. Most studies have shown that the numerical scheme of Allen-Cahn equation satisfies the maximum principle under the strictly limitation of the time and space step ratio. Could the constraint of this step ratio be removed?

The primary goal of this paper is to analyse the second-order Crank-Nicolson scheme for Allen-Cahn equation. Prove the existence of the difference schemes by Browder fixed point theorem and unconditional convergence of the difference solution through the energy analysis method by auxiliary functionals. Based on this result, it is demonstrated that the difference scheme preserves the maximum principle.

The paper is organized roughly as follows. In Sect.2, the fully discretized Crank-Nicolson scheme and the etimation of the truncation error are provided. The existence of the difference schemes is proved by using Browder fixed point theorem in Sect.3. In Sect.4, we achieve the unconditional convergence through the energy analysis method; moreover, prove that the scheme preserves the weak maximum principle. In Sect.5, a numerical example is presented to illustrate the validity and feasibility of the scheme.

2. The Discussion of the Difference Scheme

Let m, n be the positive integers. We divide equally the interval $[a,b]$ into m equal segments, the interval $[0,T]$ into n equal parts. Let $h = (b-a)/m, \tau = T/n, x_i = a + ih, 0 \le$