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A POSTERIORI ERROR ESTIMATES FOR A MODIFIED WEAK GALERKIN FINITE ELEMENT APPROXIMATION OF SECOND ORDER ELLIPTIC PROBLEMS WITH DG NORM*

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Abstract

In this paper, we derive a residual based a posteriori error estimator for a modified weak Galerkin formulation of second order elliptic problems. We prove that the error estimator used for interior penalty discontinuous Galerkin methods still gives both upper and lower bounds for the modified weak Galerkin method, though they have essentially different bilinear forms. More precisely, we prove its reliability and efficiency for the actual error measured in the standard DG norm. We further provide an improved a priori error estimate under minimal regularity assumptions on the exact solution. Numerical results are presented to verify the theoretical analysis.

Mathematics subject classification: 65N15; 65N30.
Key words: Modified weak Galerkin method, A posteriori error estimate, A medius error analysis.

1. Introduction

Weak Galerkin (WG) finite element methods (FEM) were initially proposed and analyzed in [44, 45, 56, 57] for second elliptic problems. Since then, WG methods have been extended to solving biharmonic equations [42, 48, 52, 77], linear elasticity problems [12, 55], Stokes equations [11, 58, 76], Brinkman problems [43, 64], Darcy-Stokes equations [15, 37], Helmholtz equations [24, 46, 61], Maxwell equations [49, 51, 67], Biot's consolidation model [16, 31, 69], Reissner-Mindlin plate problems [47], the Cahn-Hilliard equation [60], convection-diffusion-reaction equations [10, 39], eigenvalue problems [73–75] and other problems [32, 36, 38, 41, 68]. More recently, a primal-dual WG methods was proposed to solve second order elliptic equations

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in non-divergence form [54], and it was also extended to solving the Fokker-Planck type equations in [53]. WG methods are based on weak derivatives which allow for totally discontinuous functions of piecewise polynomials on partitions. Therefore, WG methods have many advantages similar to discontinuous Galerkin (DG) methods [3], including the property of high order of accuracy, the flexibility in handling unstructured meshes, and their suitability for hp-adaptive computations. We also refer the reader to [59] for the similarities and differences between WG methods and hybridizable discontinuous Galerkin (HDG) methods. More recently, a unified study of conforming FEM, nonconforming FEM, mixed FEM, WG methods and HDG methods is presented in [29].

In this paper, we attempt to give a posteriori error analysis for a modified weak Galerkin method for the second order problem:

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega, \\ u &= g \qquad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d (d=2,3)$ is a bounded polyhedral domain with boundary $\partial\Omega$, $f \in L^2(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$.

For simplicity, we present our analysis for the model problem (1.1) only in two dimensions. The extension to three dimensional problem can be obtained with only straightforward modifications.

The modified weak Galerkin (MWG) considered in this work was proposed in [63] to solve the problem (1.1), therein a priori error analysis was carried out detailedly. We find that similar idea was also presented in [35], but it was called dual wind DG method. By employing a new modified weak derivative, the MWG method contains a interior penalty term which is similar to interior penalty DG (IPDG) methods, but it needs not choose large parameter to satisfy the stability. Some applications of MWG methods to other problems such as parabolic equations, Sobolev equations, Stokes equations, variational inequalities and Biot's consolidation model, can be found in [26], [27], [50], [72] and [62], respectively.

A posteriori error estimates for DG methods have been extensively explored in the literature (see, for example, [1, 2, 4, 5, 8, 9, 20–22, 25, 30, 33, 40, 65, 71] and references therein). However, for WG methods, most of the existing works concentrate only on a priori error estimates, the corresponding a posteriori error analysis is very rare. The first work in this direction was by Chen, Wang and Ye in [14], where they have proposed and analyzed a residual based a posteriori error estimator for WG methods, the techniques they used rely on the Helmholtz decomposition. Later on, by employing some techniques used for IPDG methods [33], Zhang and Lin [78] derived a posterior error estimator for the MWG method. The resulting posteriori estimators obtained in [14, 78] both contain the terms $h_T\|f + \operatorname{div}(\nabla_w u_h)\|_{0,T}$ and $h_e^{1/2}\|[\![\nabla_w u_h \cdot \mathbf{n}_e]\!]\|_{0,e}$, with $\nabla_w u_h$ the weak gradient of WG finite element solution u_h. Thus, the error estimators in [14, 78] are different from the ones developed by Karakashian and Pascal for IPDG methods [33]. Since the MWG method has a interior penalty term similar to IPDG methods, one may ask a natural question: Can the error estimator used for IPDG methods [33] still gives theoretical upper and lower bounds for the MWG method? The answer to this question is not straightforward because the MWG method has itself special bilinear form. The first contribution of our work is to answer this question in the affirmative. To this end, we first define the error estimator as follows:

$$\eta_h^2 = \sum_{T\in\mathcal{T}_h} \eta_T^2 + \sum_{e\in\mathcal{E}_h} \eta_{e,1}^2 + \sum_{e\in\mathcal{E}_h^I} \eta_{e,2}^2, \tag{1.2}$$