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PHYSICS INFORMED NEURAL NETWORKS (PINNs) FOR APPROXIMATING NONLINEAR DISPERSIVE PDEs*

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Abstract

We propose a novel algorithm, based on physics-informed neural networks (PINNs) to efficiently approximate solutions of nonlinear dispersive PDEs such as the KdV-Kawahara, Camassa-Holm and Benjamin-Ono equations. The stability of solutions of these dispersive PDEs is leveraged to prove rigorous bounds on the resulting error. We present several numerical experiments to demonstrate that PINNs can approximate solutions of these dispersive PDEs very accurately.

Mathematics subject classification: 35Q53, 65M12, 65M15.
Key words: Nonlinear dispersive PDEs, Deep learning, Physics Informed Neural Networks.

1. Introduction

Deep learning i.e., the use of deep neural networks for regression and classification, has been very successful in many different contexts in science and engineering [30]. These include image analysis, natural language understanding, game intelligence and protein folding. As deep neural networks are universal function approximators, it is natural to employ them as ansatz spaces for solutions of ordinary and partial differential equations, paving the way for their successful use in scientific computing. A very incomplete list of examples where deep learning is used for the numerical solutions of differential equations includes the solution of high-dimensional linear and semi-linear parabolic partial differential equations [10, 14] and references therein, and for many-query problems such as those arising in uncertainty quantification (UQ), PDE constrained optimization and (Bayesian) inverse problems. Such problems can be recast as parametric partial differential equations and the use of deep neural networks in their solution is explored for elliptic and parabolic PDEs in [23,44], for transport PDEs [25] and for hyperbolic and related PDEs [6,35–37], and as operator learning frameworks in [2,29,31,33] and references therein. All the afore-mentioned methods are of the *supervised learning* type [13] i.e., the underlying deep neural networks have to be trained on *data*, either available from measurements or generated by numerical simulations.

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However, there are several interesting problems for PDEs where generating training data might be very expensive. A different strategy might be relevant for such problems, namely the so-called *Physics informed neural networks* (PINNs) which collocate the PDE residual on *training points* of the approximating deep neural network, thus obviating the need for generating training data. Proposed originally in [7, 26, 27], PINNs have been revived and developed in significantly greater detail recently in the pioneering contributions of Karniadakis and collaborators. PINNs have been successfully applied to simulate a variety of forward and inverse problems for PDEs, see [5, 18, 19, 32, 34, 38, 41, 45, 50–52, 55] and references therein.

In a recent paper [40], the authors obtain rigorous estimates on the error due to PINNs for the forward problem for a variety of linear and non-linear PDEs, see [39] for similar results on inverse problems and [54] for a different perspective on error estimates for PINNs. Following [40], one can expect that PINNs could be efficient at approximating solutions of nonlinear PDEs as long as classical solutions to such PDEs exist and are *stable* in a suitable sense. So far, PINNs have only been proposed and tested for a very small fraction of PDEs. It is quite natural to examine whether they can be efficient at approximating other types of PDEs and in particular, if the considerations of [40] apply to these PDEs, then can one derive rigorous error estimates for PINNs?

In this paper, we investigate the utility of PINNs for approximately a large class of PDEs which arises in physics i.e., non-linear dispersive equations that model different aspects of shallow water waves [28]. These include the famous Korteweg-De Vries (KdV) equation and its high-order extension, the so-called Kawahara equation, the well-known Camassa-Holm type equations and the Benjamin-Ono equations. All these PDEs have several common features, namely

- They model dispersive effects in shallow-water waves.
- The interesting dynamics of these equations results from a balance between non-linearity and dispersion.
- They are completely integrable and contain interesting structures such as interacting solitons in their solutions.
- Classical solutions and their stability have been extensively investigated for these equations.
- Standard numerical methods, such as finite-difference [4,8,15,16,21,53] and finite-element [9, 22] for approximating these equations can be very expensive computationally. In particular, it can be very costly to obtain low errors due to the high-order (or non-local) derivatives in these equations leading to either very small time-steps for explicit methods or expensive non-linear (or linear) solvers for implicit methods.

Given these considerations, it is very appealing to investigate if PINNs can be successfully applied for efficiently approximating these nonlinear dispersive PDEs. To this end, we adapt the PINNs algorithm to this context in this paper and prove error estimates for PINNs, leveraging the stability of underlying classical solutions into error bounds. Moreover, we perform several numerical experiments for the KdV, Kawahara, generalized Camassa-Holm and Benjamin-Ono equations to ascertain that PINNs can indeed approximate dispersive equations to high-accuracy, at low computational cost.