Journal of Computational Mathematics
Vol.40, No.1, 2022, 127–146.
http://www.global-sci.org/jcm
doi:10.4208/jcm.2007-m2020-0164

UNCONDITIONALLY OPTIMAL ERROR ESTIMATES OF THE BILINEAR-CONSTANT SCHEME FOR TIME-DEPENDENT NAVIER-STOKES EQUATIONS*

Huaijun Yang and Dongyang Shi[1)]
School of Mathematics, Zhengzhou University of Aeronautics, Zhengzhou 450046, China
School of Mathematics and Statistics, Zhengzhou University, Zhengzhou 450001, China
Email: huaijunyang@zua.edu.cn, shi_dy@zzu.edu.cn

Abstract

In this paper, the unconditional error estimates are presented for the time-dependent Navier-Stokes equations by the bilinear-constant scheme. The corresponding optimal error estimates for the velocity and the pressure are derived unconditionally, while the previous works require certain time-step restrictions. The analysis is based on an iterated time-discrete system, with which the error function is split into a temporal error and a spatial error. The τ-independent (τ is the time stepsize) error estimate between the numerical solution and the solution of the time-discrete system is proven by a rigorous analysis, which implies that the numerical solution in L^∞-norm is bounded. Thus optimal error estimates can be obtained in a traditional way. Numerical results are provided to confirm the theoretical analysis.

Mathematics subject classification: 65M60, 65M12, 65N15.
Key words: Navier-Stokes equations, Unconditionally optimal error estimates, Bilinear-constant scheme, Time-discrete system.

1. Introduction

In this paper, we pay our attention to the following time-dependent incompressible Navier-Stokes equation in two dimensions:

$$\boldsymbol{u}_t - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f}, \quad (\boldsymbol{x},t)\in\Omega\times(0,T], \tag{1.1}$$
$$\nabla\cdot\boldsymbol{u} = 0, \quad (\boldsymbol{x},t)\in\Omega\times(0,T], \tag{1.2}$$
$$\boldsymbol{u}(\boldsymbol{x},t) = 0, \quad (\boldsymbol{x},t)\in\partial\Omega\times(0,T], \tag{1.3}$$
$$\boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}), \quad \boldsymbol{x}\in\Omega, \tag{1.4}$$

where $\Omega\subset\mathbb{R}^2$ is a rectangular domain with boundary $\partial\Omega$ and $\boldsymbol{x}=(x_1,x_2)$. $\boldsymbol{u}=(u_1,u_2)$ represents the velocity vector, p the pressure, $\boldsymbol{f}=(f_1,f_2)$ the body force, $\nu=1/Re$ the viscosity coefficient and Re is the Reynolds number.

It is well known that the time-dependent incompressible Navier-Stokes equations is a very important system in the mathematical physics and the fluid mechanics fields. In the past several decades, a lot of efforts have been devoted to the development of efficient numerical approximations for solving this system [1–9]. In particular, a new fully-discrete finite element

* Received June 19, 2020 / Accepted July 23, 2020 /
Published online February 27, 2021 /
1) Corresponding author

nonlinear Galerkin method was studied to the long time integration of the Navier-Stokes equations in [5] through the spatial discretization based on two-grid finite element technique and the time discretization based on Euler explicit scheme with variable time stepsize. However, a certain time step constraint was required so as to obtain the boundedness and convergence of the above method. In [6], the Lagrange-Galerkin mixed finite element approximation of the Navier-Stokes equations was discussed and the corresponding optimal estimate was derived with the time stepsize restriction $\tau = \mathcal{O}(h^\sigma)$, where $\sigma > (n-1)/2$ and n denotes the dimensions of the domain Ω. A suboptimal convergence rate $\mathcal{O}(h + \tau + h^2/\tau)$ was derived in [8] by a characteristics type Galerkin finite element methods. The restriction could become more serious when the problem was considered in a high-dimensional space and/or with a non-uniform mesh.

On the other hand, for the nonlinear problems, linearized (semi)-implicit schemes are more efficient since at each time step, the schemes only require solving the linear systems. However, the time step restriction condition of the linearized schemes arising from the error analysis is always a crucial issue (see [10]- [14]). In addition, the L^∞ boundness of the numerical solution is an essential condition in the error analysis. Most previous works require certain time step restrictions when the inverse inequality is used to bound the numerical solution. Therefore, there have been some attempts to reduce the time step restriction conditions. For example, a new approach was introduced in [15] and [16] to get unconditional stability and optimal error estimates of a linearized backward Euler Galerkin/Galerkin-mixed finite element methods for the time-dependent Joule heating equations and the incompressible miscible flow in porous media, respectively. This new approach is based on a new error splitting technique by a corresponding time-discrete system. Then, with the proved certain regularity of the solution of the time-discrete system, it follows that

$$\|\boldsymbol{U}_h^n\|_{0,\infty} \le \|R_h \boldsymbol{U}^n\|_{0,\infty} + \|R_h \boldsymbol{U}^n - \boldsymbol{U}_h^n\|_{0,\infty} \le C + Ch^{-d/2}h^{r+1},$$

where $\boldsymbol{U}_h^n$ is the finite element solution, R_h is the Galerkin projection, d is the dimensions of Ω and r is the degree of piecewise polynomial. Thus, the boundedness of the numerical solution $\boldsymbol{U}_h$ in L^∞-norm can be derived without any time step restriction. Subsequently, this approach has been applied to many other problems [17]- [22] to study the convergence or superconvergence of the numerical schemes and to deduce the error estimates almost unconditionally (i.e., the step sizes $h,\ \tau \le s_0$ for some small positive constant s_0). Moreover, in [23], the unconditional stability and error estimates of modified characteristics finite element methods were researched for the time-dependent Navier-Stokes equations. However, the boundary $\partial\Omega$ of Ω should belong to C^2 due to the boundedness of numerical solution $\boldsymbol{U}_h^n$ in $W^{1,\infty}$-norm used in the error analysis.

In this paper, the unconditionally optimal error estimates are investigated for the time-dependent Navier-Stokes equations with Lipschitz boundary $\partial\Omega$, which is weaker than that in [23]. The spatial discretization is approximated by a low order conforming bilinear-constant mixed finite element method [24, 25], and the time discretization is approximated by semi-implicit Euler scheme. The analysis is based on an error splitting technique proposed in [15, 16] with a time-discrete system. More precisely, the τ-independent error estimate is first derived for the time-discrete system, then the numerical solution in L^∞-norm can be bounded in terms of the mathematical induction and inverse inequality, which lead to the unconditionally optimal error estimates are achieved in a routine way.

The rest of this paper is organized as follows. In section 2, we introduce some notations and preliminaries. Moreover, we also present the linearized semi-implicit Euler Galerkin scheme and the main results. The temporal and the spatial error estimates are established in section