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A FINITE ELEMENT ALGORITHM FOR NEMATIC LIQUID CRYSTAL FLOW BASED ON THE GAUGE-UZAWA METHOD*

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Abstract

In this paper, we present a finite element algorithm for the time-dependent nematic liquid crystal flow based on the Gauge-Uzawa method. This algorithm combines the Gauge and Uzawa methods within a finite element variational formulation, which is a fully discrete projection type algorithm, whereas many projection methods have been studied without space discretization. Besides, error estimates for velocity and molecular orientation of the nematic liquid crystal flow are shown. Finally, numerical results are given to show that the presented algorithm is reliable and confirm the theoretical analysis.

Mathematics subject classification: 65M15, 65M60.
Key words: Nematic liquid crystal model, Finite element approximation, Gauge-Uzawa method, Error analysis.

1. Introduction

Given a bounded and convex domain $\Omega \subset \mathbb{R}^2$, we consider the following hydrodynamics system modeling the flow of nematic liquid crystal material [1, 18]

$$\mathbf{u}_t - \mu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p + \lambda\nabla\cdot(\nabla\mathbf{b}\odot\nabla\mathbf{b}) = \mathbf{f}, \tag{1.1a}$$

$$\mathbf{b}_t - \gamma\Delta\mathbf{b} + (\mathbf{u}\cdot\nabla)\mathbf{b} - \gamma|\nabla\mathbf{b}|^2\mathbf{b} = 0, \tag{1.1b}$$

$$\nabla\cdot\mathbf{u} = 0, \quad |\mathbf{b}| = 1, \tag{1.1c}$$

for $(x,t)\in Q_T$, where $Q_T = \Omega\times(0,T)$ with a fixed $T\in(0,\infty)$. Here, $\mathbf{u}(x,t): Q_T \to \mathbb{R}^2$ and $p(x,t): Q_T \to \mathbb{R}$ denote the velocity field and the pressure of the flow, respectively. Besides, $\mathbf{b}(x,t): Q_T \to \mathbb{S}$ is the director, which represents the molecular orientation field of the nematic liquid crystal material and describes the average molecular alignment, where $\mathbb{S}\subset\mathbb{R}^2$ is a unit circle. In addition, $\mathbf{f}(x,t): Q_T \to \mathbb{R}^2$ represents a body force on the flow. Three parameters μ, λ and γ denote the kinematic viscosity, the competition between kinetic and potential energy, and the microscopic elastic relaxation time for the molecular orientation field, respectively. Hereafter, $|\nabla\mathbf{b}|$ or $|\mathbf{b}|$ denotes the Euclidean norm of $\nabla\mathbf{b}$ or $\mathbf{b}$. $\nabla\mathbf{b}\odot\nabla\mathbf{b}$ is an 2×2 matrix

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whose (i,j)-the entry is written by $(\sum_{k=1}^{2} \frac{\partial b_k}{\partial x_i} \frac{\partial b_k}{\partial x_j})_{i,j}$. As in [1], in this paper the system (1.1) is considered in conjunction with the following initial and boundary conditions:

$$\begin{aligned} &\mathbf{u}(x,0) = \mathbf{u}_0(x), \quad \mathbf{b}(x,0) = \mathbf{b}_0(x), \quad \forall x \in \Omega, \\ &\mathbf{u}|_{S_T} = 0, \quad \partial_{\mathbf{n}} \mathbf{b}|_{S_T} = 0, \end{aligned} \tag{1.2}$$

with $\nabla \cdot \mathbf{u}_0 = 0$ and $|\mathbf{b}_0| = 1$, where $S_T = \partial\Omega \times (0,T)$ and $\mathbf{n}$ is the outer unit normal of $\partial\Omega$.

The Ericksen-Leslie model, established by Ericksen [9, 10] and Leslie [16], can simulate the hydrodynamics of the nematic liquid crystal flows, and it is the macroscopic continuum description of the time evolution of the nematic liquid crystal materials under the influence of both flow velocity and molecular orientation. The system (1.1) was derived by Lin [18] initially as a simplified form of the Ericksen-Leslie model, which is a system of the Navier-Stokes equations coupled with a convective harmonic map heat flow equation to govern the dynamics of the director field. Although this system neglects the Leslie stress in the Ericksen-Leslie model, it still retains some essential difficulties of the Ericksen-Leslie model and keeps the core of the mathematical structure, such as strong nonlinearities and constraints, as well as the physical structure, such as the anisotropic effect of elasticity on the velocity field. Thus, the system (1.1) (the simplified Ericksen-Leslie model) can be regarded as a nice initial step towards the theoretical and numerical analysis of the original problem (the Ericksen-Leslie model) [2].

Because of the mathematical and engineering importance of the system (1.1), the simplified Ericksen-Leslie system, there are numerous papers devoted to the theoretical analysis of this system, such as the existence, uniqueness and regularity of solutions. The local existence and uniqueness of strong solutions have been proved by using the standard energy method with rough data [32]. In [15], a blow up criterion has been established for the short time classical solution of the simplified version of Ericksen-Leslie system in two and three dimensions. Hong [14] has given a global existence of solutions to a simplified model of the Ericksen-Leslie system in two dimension with initial data, where the solutions are constructed with at most a finite number of singular times. Both interior and boundary regularity theorems for this system under smallness conditions have been established [19]. Then, the authors have given the existence of global (in time) weak solutions on a bounded smooth domain, which are smooth everywhere with possible exceptions of finitely many singular times. Besides, the global regularity and uniqueness of weak solutions are proved by Xu and Zhang [34] and Lin and Wang [23], respectively.

Since the governing equations (1.1) of the simplified Ericksen-Leslie model include not only the incompressibility, the strong nonlinearity and the physical and nonconvex side constraint $|\mathbf{b}| = 1$, but also the coupling between the harmonic map flow and the fluid equations of motion, which make it not easy to solve these equations effectively. Therefore, much effort has been throwing to the development of some efficient numerical methods for investigating this problem.

On one hand, in order to weaken the nonconvex side constraint, a well-known penalty formulation for (1.1) called the Ginzburg-Landau function is studied by Lin and Liu [20]. In fact, Liu and Walkington [21] initially considered the numerical approximation of this Ginzburg-Landau penalized problem in two-dimensional domains. Further, to eliminate the need of Hermite finite elements, the same authors have constructed some mixed approximations where derivatives of the director field are approximated "independently" of the director [22]. A fully discrete mixed scheme, based on continuous finite elements in space and a linear semi-implicit first-order integration in time, has been shown in [12]. Besides, Cabrales et al. [6, 7] have presented a projection-based time-splitting algorithm, where the velocity and pressure are computed by using a projection-based algorithm and the director is computed jointly to an auxiliary vari-