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ELLIPTIC RECONSTRUCTION AND A POSTERIORI ERROR ESTIMATES FOR FULLY DISCRETE SEMILINEAR PARABOLIC OPTIMAL CONTROL PROBLEMS*

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Abstract

This article studies a posteriori error analysis of fully discrete finite element approximations for semilinear parabolic optimal control problems. Based on elliptic reconstruction approach introduced earlier by Makridakis and Nochetto [25], a residual based a posteriori error estimators for the state, co-state and control variables are derived. The space discretization of the state and co-state variables is done by using the piecewise linear and continuous finite elements, whereas the piecewise constant functions are employed for the control variable. The temporal discretization is based on the backward Euler method. We derive a posteriori error estimates for the state, co-state and control variables in the $L^\infty(0,T;L^2(\Omega))$-norm. Finally, a numerical experiment is performed to illustrate the performance of the derived estimators.

Mathematics subject classification: 49J20, 65J15, 65N30.
Key words: Semilinear parabolic optimal control problem, Finite element method, The backward Euler method, Elliptic reconstruction, A posteriori error estimates.

1. Introduction

Let Ω be a bounded convex polygonal domain in $\mathbb{R}^d\,(d\le 3)$ with Lipschitz boundary $\partial\Omega$. Set $\Omega_T=\Omega\times(0,T]$, $\Gamma_T=\partial\Omega\times[0,T]$ with $T<\infty$. We shall consider the following semilinear parabolic optimal control problems:

$$\min_{u\in U_{ad}} J(y,u)=\min_{u\in U_{ad}}\frac{1}{2}\int_0^T\left\{\|y-y_d\|^2+\|u\|^2\right\}dt \tag{1.1}$$

subject to the state equation

$$\begin{cases}\dfrac{\partial}{\partial t}y-\Delta y+\phi(y)=f+u, & \text{in } \Omega_T,\\ y(x,0)=y_0(x), & \text{in } \Omega,\\ y=0, & \text{on } \Gamma_T,\end{cases} \tag{1.2}$$

and the control constraints

$$u_a\le u(x,t)\le u_b \quad \text{a.e.} \quad \text{in } \Omega_T, \tag{1.3}$$

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where $y = y(x,t)$ and $u = u(x,t)$ denote the state variable and the control variable, respectively. Moreover, the initial function $y = y_0(x)$ and the forcing function $f = f(x,t)$ are assumed to be smooth in their respective domain of definitions. The set of admissible controls is defined by

$$U_{ad} = \Big\{ u \in L^2\big(0,T; L^2(\Omega)\big) : u_a \leq u \leq u_b \quad \text{a.e.} \quad \text{in } \Omega_T \Big\}$$

with $u_a, u_b \in \mathbb{R}$ fulfill $u_a < u_b$. In the above, $\phi(\cdot)$ is of class C^2 with respect to the state variable. For any $R > 0$ the function $\phi(\cdot) \in W^{2,\infty}(-R,R)$, $\phi'(y) \in L^2(\Omega)$ for any $y \in L^2(0,T; H_0^1(\Omega))$, and $\phi'(y) \geq 0$. Moreover, we assume that there exists positive constants C_L and K_L such that, for all $y_1, y_2 \in L^2(0,T; H^1(\Omega))$,

$$\big|\phi(y_1) - \phi(y_2)\big| \leq C_L |y_1 - y_2|, \quad \big|\phi'(y_1) - \phi'(y_2)\big| \leq K_L |y_1 - y_2|. \tag{1.4}$$

The numerical treatment of optimal control problems with time-dependent control is of paramount importance because of their various applications in science and engineering (cf. [20, 26, 33]). The finite element method is widely used numerical method in computing optimal control problems, see [1, 7, 14, 17, 19, 22, 26, 33] and references quoted therein. Though theory of a posteriori error analysis of finite element methods for elliptic control problems is well-developed, the literature seems lack for time dependent semilinear and nonlinear control problems. For the standard parabolic problems, we refer the reader to [34,35] for the space-time adaptivity, [13, 28] for only time adaptivity and [3–5, 9] for spatial adaptivity keeping temporal variable continuous. In space-time adaptivity, the finite element discretization is based on the space-time variational formulation and the error indicators include both space and time errors. Makridakis and Nochetto [25] have used energy techniques in conjunction with an appropriate pointwise representation of the error based on an elliptic reconstruction operator to recover the optimal a posteriori error estimates in the $L^\infty(0,T; L^2(\Omega))$-norm for the linear parabolic problem. The role of the elliptic reconstruction operator in a posteriori estimates is quite similar to the role played by elliptic projection introduced by Wheeler [35] for recovering optimal a priori error estimates for finite element method to parabolic problems. Essentially, using the elliptic reconstruction $\tilde{z}$, the total error $e = z_h - z$, where z_h is the finite element solution, can be split into $z_h - \tilde{z}$, the error due to elliptic reconstruction and $\tilde{z} - z$, which is the difference between elliptic reconstruction and the exact solution. The construction of $\tilde{z}$ is such that the estimates of $z_h - \tilde{z}$ are based on a posteriori analysis of the elliptic problem, and the estimate for $\tilde{z} - z$ is found out using the standard energy argument in terms of the estimates of $z_h - \tilde{z}$. The fully discrete a posteriori error estimates for parabolic problem using elliptic reconstruction can be found in [16], for maximum norm estimates in [8], and for discontinuous Galerkin methods in [10]. The previous work on the a posteriori error analysis for semilinear parabolic problems are described in [2, 15, 27, 36], and for the semilinear parabolic interface problem, see [31, 32].

Some comparable work on nonlinear parabolic optimal control problems can be found in [11,18,23,24,29]. In [11], the authors focus their work on three types of optimization problems namely, the von Kfirmfin plate equations of nonlinear elasticity, the Ginzburg-Landau equations of super conductivity, and the Navier-Stokes equations for incompressible viscous flows. First, they have studied the existence of optimal solution for a class of nonlinear control problems with some appropriate assumptions and have derived optimality system using Lagrange multipliers technique. Further, a priori type error estimates for the approximate control problem is also discussed. In [18], for fully discrete control problem, Li *et al.* have used spectral approximation scheme for the space discretization and the backward Euler scheme for the time discretization.