Journal of Computational Mathematics
Vol.40, No.2, 2022, 177–204.

http://www.global-sci.org/jcm
doi:10.4208/jcm.2010-m2019-0200

CONVERGENCE AND MEAN-SQUARE STABILITY OF EXPONENTIAL EULER METHOD FOR SEMI-LINEAR STOCHASTIC DELAY INTEGRO-DIFFERENTIAL EQUATIONS*

Haiyan Yuan
Department of Mathematics, Heilongjiang Institute of Technology, Harbin 158100, China
Email: yhy82_47@163.com

Abstract

In this paper, the numerical methods for semi-linear stochastic delay integro-differential equations are studied. The uniqueness, existence and stability of analytic solutions of semi-linear stochastic delay integro-differential equations are studied and some suitable conditions for the mean-square stability of the analytic solutions are also obtained. Then the numerical approximation of exponential Euler method for semi-linear stochastic delay integro-differential equations is constructed and the convergence and the stability of the numerical method are studied. It is proved that the exponential Euler method is convergent with strong order $\frac{1}{2}$ and can keep the mean-square exponential stability of the analytical solutions under some restrictions on the step size. In addition, numerical experiments are presented to confirm the theoretical results.

Mathematics subject classification: 60H35, 65C20, 65C30, 65L20.
Key words: Semi-linear stochastic delay integro-differential equation, Exponential Euler method, Mean-square exponential stability, Trapezoidal rule.

1. Introduction

Delay integro-differential equations (DIDEs) are often used to model some problems in biology, medicine, and many other fields. Here we highlight [13] for models in population dynamics and [12] for applications in physics, engineering or economy. Taking random noise into account, we can obtain stochastic delay integro-differential equations (SDIDEs). Stochastic delay integro-differential equations can be viewed as the generalizations of deterministic DIDEs and stochastic delay differential equations (SDDEs). Explicit solutions can hardly be obtained for SDIDEs, thus it is necessary to develop numerical methods and study their properties. Many mathematicians have devoted their effort to study them and have obtained a large number of achievements. We refer the numerical analysis linear SDIDEs to Shaikhet and Roberts [35], Ding, et al. [5], Rathinasamy and Balachandran [33] and Jiang, et al. [11], and the numerical analysis nonlinear SDIDEs to Li and Gan [18], Mirzaee and Hadadiyan [29] and Mirzaee, and Samadyar [30].

The phenomenon of stiffness appears in the process of applying a certain numerical method to ODEs and SDEs. It is known that the stiffness makes standard explicit integrators useless. Nevertheless, the implicit scheme does not perform well either for the step size reduction is forced by accuracy requirements: the method tends to resolve all the oscillations in the solutions, hence its numerical inefficiency. Due to the cost of computing the Jacobian, and the

* Received September 11, 2019 / Revised version received April 26, 2020 / Accepted October 19, 2020 / Published online August 30, 2021 /

exponential or related function of Jacobian, many works are directed at the semi-linear problems. Exponential integrators perform useful in solving semi-linear problems because they can solve exactly the linear part. In order to overcome the stiffness in the semi-linear problems, Lawson [17] firstly combines the exponential function with explicit Runge-Kutta methods and gives exponential Runge-Kutta methods. The stability properties of exponential Runge-Kutta methods are investigated in [8,21,31]. In [9,19,20,27,28,32], exponential Runge-Kutta methods, exponential multi-step methods, exponential Rosenbrock methods and exponential general linear methods for semi-linear problems have been studied. For a detailed overview on the history of exponential integrators as well as recent achievements, we refer to [24] and [10]. As there are quite a few results about exponential integrators for semi-linear SDDEs and semi-linear SIDEs, it is natural to apply exponential integrators to semi-linear stochastic delay integro-differential equations (SLSDIDEs) and consider whether they have good stability and convergence properties.

In this paper, we focus on the SLSDIDEs with the following form

$$\begin{cases} dy(t) = (Ay(t) + f\Big(t,\ y(t),\ y(t-\tau), \displaystyle\int_{t-\tau}^{t} k(t,s,y(s))ds\Big)dt \\ \qquad\qquad + g\Big(t, y(t), y(t-\tau), \displaystyle\int_{t-\tau}^{t} k(t,s,y(s))ds\Big)d\omega(t), \qquad t \in [0,T], \\ y(t) = \phi(t), \qquad t \in [-\tau, 0], \end{cases} \tag{1.1}$$

where the delay parameter τ is a positive constant, $A \in \mathbb{R}^{d\times d}$. $f : \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}^d$, $g : \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}^{d\times r}$ are continuous functions, $k : \mathbb{R}\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^d$ is a second order continuous differentiable function which satisfies that for all $t \ge 0$, $t-\tau \le s \le t$, $k(t,s,0) = 0$ and k'' is bounded, that is for any $y(s)$, there is a positive constant D_0 such that $|k''| \le D_0$. The initial data ϕ has bounded moments, that is, for each $p > 0$, there is a finite positive constant M_p such that $E\,|\phi|^p < M_p$. $\omega(t)$ is an $r-$dimensional Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge0}, \mathrm{P})$ with a filtration $\{\mathcal{F}_t\}_{t\ge0}$ satisfying the usual conditions (that is, it is increasing and right continuous with $\mathcal{F}_0$ containing all P-null sets).

Unless otherwise specified, let $|x|$ be the Euclidean norm in $x \in \mathbb{R}^d$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{trace(A^TA)}$. For simplicity, we also denote $a\wedge b = \min\{a,b\}$, $a \vee b = \max\{a,b\}$.

To ensure the existence and uniqueness of the solutions, we assume that f and g satisfy the following Lipschitz condition

(1) (Lipschitz condition) There exists a positive constant L_1, for all $x_1, y_1, z_1, x_2, y_2, z_2 \in \mathbb{R}^d$, $t \ge 0$, such that

$$\begin{aligned} &|f(t,x_1,y_1,z_1) - f(t,x_2,y_2,z_2)|^2 \vee |g(t,x_1,y_1,z_1) - g(t,x_2,y_2,z_2)|^2 \\ \le& L_1\Big(\,|x_1-x_2|^2 + |y_1-y_2|^2 + |z_1-z_2|^2\,\Big). \end{aligned} \tag{1.2}$$

It has been shown in [37] that the global Lipschitz condition (1.2) implies the following linear growth condition.

(2) (Linear growth condition) There exists a positive constant L_2, for any $x, y, z \in \mathbb{R}^d$, $t \ge 0$, such that

$$|f(t,x,y,z)|^2 \vee |g(t,x,y,z)|^2 \le L_2\Big(1 + |x|^2 + |y|^2 + |z|^2\Big). \tag{1.3}$$