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# CONSTRUCTION OF CUBATURE FORMULAS VIA BIVARIATE QUADRATIC SPLINE SPACES OVER NON-UNIFORM TYPE-2 TRIANGULATION*

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**Abstract**

In this paper, matrix representations of the best spline quasi-interpolating operator over triangular sub-domains in $S^1_2(\Delta^{(2)}_{mn})$, and coefficients of splines in terms of B-net are reviewed firstly. Moreover, by means of coefficients in terms of B-net, computation of bivariate numerical cubature over triangular sub-domains with respect to variables $x$ and $y$ is transferred into summation of coefficients of splines in terms of B-net. Thus concise bivariate cubature formulas are constructed over rectangular sub-domain. Furthermore, by means of module of continuity and max-norms, error estimates for cubature formulas are derived over both sub-domains and the domain.

*Mathematics subject classification:* 65D07, 65D32.
*Key words:* Multivariate spline, Bivariate cubature, Conformality of Smoothing Cofactor Method, B-net, Non-uniform Type-2 Triangulation.

## 1. Introduction

As we know, there are always several complicated and important ways to generalize one-dimensional problems to multi-dimensional ones. This is the case for multivariate splines, where three main approaches have been innovated and developed, namely, conformality of smoothing cofactor method [21, 22], Cartesian tensor product multivariate splines [18, 20], and B-net method [8, 9] in chronological order. Among them, the most general approach to study multivariate splines is the conformality of smoothing cofactor method. The reason may lie in its ability of determining dimensions of multivariate splines over arbitrary partition, spline bases with minimal support, multivariate interpolation, and spline quasi-interpolation,etc. [7, 22].

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Specifically, we denote by $S_k^\mu(\Delta)$ the multivariate spline space with degree $k$ and smoothness $\mu$ over the domain $D$ with respect to the partition $\Delta$. The study showed that the dimension depended heavily on the geometry of the partition $\Delta$ [19,26]. Using the valuable conformality of smoothing cofactor method yielded bivariate cubic [12] and quartic [23] spline spaces on uniform type-2 triangulation, and spline quasi-interpolating operators which reproduced some bivariate polynomials with different degrees. Bivariate quadratic splines and spline quasi-interpolating operators over non-uniform type-2 triangulation were investigated in [24, 25]. After that, some new results and applications are obtained in bivariate quadratic spline space $S_2^1(\Delta_{mn}^{(2)})$, including approximation of derivatives of the operators [3] and numerical integration [4,10]. Bivariate cubic spline spaces and spline quasi-interpolation have been studied [15, 16], and have approximation of derivatives of these operators [14]. Recently, the application in solving partial differential equations with spline quasi-interpolation operators has attracted people's attention. Spline quasi-interpolating projectors were used on a bounded interval for the numerical solution of linear Fredholm integral equations of the second kind in [1,5]. Optimal super-convergent quasi-interpolants over type-2 triangulation were considered to be as spline methods for some integral equations [6]. A kind of bivariate quartic splines was applied in solving linear hyperbolic equations as spline finite elements [17].

To our best knowledge, however, specific bivariate cubature formulas via $S_2^1(\Delta_{mn}^{(2)})$ over nonuniform type-2 triangulation have not been established with coefficients of splines in terms of B-net, while what was developed in [10] is numerical integration based on special B splines without data points outside the rectangular domain. Actually, by adopting coefficients in terms of B-net, we transfer the complicated direct bivariate integral computation over sub-domains with respect to integral variables $x$ and $y$ into the summation of coefficients of splines in terms of B-net. Thus we can apply them in solving partial differential equations conveniently, which will be developed in another paper due to space limitations.

A brief outline of the paper is as follows. In Section 2, we shall review some useful results of bivariate quadratic spline space $S_2^1(\Delta_{mn}^{(2)})$ presented [13], including specific matrix representations of the best spline quasi-interpolating operator and coefficients of splines in terms of B-net. Then we shall establish cubature formulas over triangular sub-domains and rectangular sub-domains based upon these coefficients of splines in terms of B-net in Section 3. At last, we shall derive error estimates for cubature formulas by using symmetric local estimates for the best spline quasi-interpolating operator we got [13] in Section 4.

## 2. Review of Explicit Matrix Representations of the Spline Quasi-Interpolation and Coefficients of Splines in Terms of B-net

The paper is considering how to construct the cubature formulas over the bounded rectangular domain based upon bivariate quadratic spline space $S_2^1(\Delta_{mn}^{(2)})$. However, the difficulties lie in computing the explicit cubature formula over each triangular sub-domain because of the different representation of bivariate quadratic splines. Hence, we shall review the explicit matrix representations of the spline quasi-interpolation over triangular sub-domains and the corresponding rectangular sub-domain in [13] in the following theorems in the section. Moreover, for the sake of illustration, we shall convert the cubature formulas into the summation of the coefficients of spines in terms of B-net over 24 triangular sub-domain in the local octagonal support in Fig.2.1. Of course, theses coefficients in terms of B-net can be obtained in [13, 22].

Denote by $\Omega := [a, b] \times [c, d]$ the rectangular domain, and by $\Omega_{ij} := [x_i, x_{i+1}] \times [y_j, y_{j+1}] \in \Omega$