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ON DISTRIBUTED H^1 SHAPE GRADIENT FLOWS IN OPTIMAL SHAPE DESIGN OF STOKES FLOWS: CONVERGENCE ANALYSIS AND NUMERICAL APPLICATIONS*

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Abstract

We consider optimal shape design in Stokes flow using H^1 shape gradient flows based on the distributed Eulerian derivatives. MINI element is used for discretizations of Stokes equation and Galerkin finite element is used for discretizations of distributed and boundary H^1 shape gradient flows. Convergence analysis with a priori error estimates is provided under general and different regularity assumptions. We investigate the performances of shape gradient descent algorithms for energy dissipation minimization and obstacle flow. Numerical comparisons in 2D and 3D show that the distributed H^1 shape gradient flow is more accurate than the popular boundary type. The corresponding distributed shape gradient algorithm is more effective.

Mathematics subject classification: 49Q10, 49Q12, 76D55, 76M10.

Key words: Shape optimization, Stokes equation, Distributed shape gradient, Finite element, MINI element, Eulerian derivative.

1. Introduction

Shape optimization of flows in computational fluid mechanics has wide applications in engineering design [6, 7, 19–21, 24]. Many numerical methods have been developed for solving these optimization and control problems by virtue of computer simulations. Numerical methods based on boundary variations can perform shape changes during shape evolutions [24, 27]. Topology optimization approaches and techniques such as level set methods can perform simultaneously shape and topological changes during evolutions, especially useful for structure optimization (see e.g. [1]). Both shape and topology optimization approaches require "shape sensitivity analysis" in shape gradient based algorithms for numerical simulations. Shape sensitivity analysis can be performed with shape calculus. The Eulerian derivative of a shape functional can be written in a distributed or boundary type formulation [6, 24], the latter of which has caused much attention due to its attractively concise appearance. As a vector distribution depending on the choice of the topological vector space (pp. 479 [6]), the shape gradient can usually be obtained by using the boundary type Eulerian derivative in the structure theorem (cf. Theorem 2.27 [24]). Most existing research works on numerical shape design algorithms typically approximate the Eulerian derivatives using the popular boundary formulation both in shape optimization [21, 26] and topology optimization methods [3, 17]. But this boundary Eulerian derivative approximated by finite elements is not suitable to be used since it fails to hold when

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the boundary is not regular enough. The distributed Eulerian derivative holds more generally [6] and thus deserves much attention in numerical computation.

The distributed Eulerian derivatives have been used for numerical shape optimization algorithms in many fields, e.g., electrical impedance tomography [17], parabolic diffusion inverse problems [23], Maxwell equation [9, 15, 22], structure design [3]. Distributed shape gradient algorithms have been shown attractive, since they have better influences on the finite element mesh quality and are usually more efficient. The advantages were shown by comparing with surface shape gradients in eigenvalue optimization [27,29] and shape reconstruction in flows [18].

Shape evolutions can be performed with L^2 shape gradient flows in the L^2 space according to the boundary formulation of Eulerian derivative. The L^2 flows may be not smooth enough for robust shape changes. H^1 shape gradient flows are useful and more popular in literature to replace L^2 shape gradient flows for performing domain deformations and ensuring real gradient descent in shape gradient algorithms (see, e.g., [18, 22, 27]). For H^1 flows, the mesh grids in the domain and on the boundary are moved simultaneously to form a new domain. Moreover, the velocity fields can be regularized in the H^1 Sobolev space. The distributed formulations of L^2 gradient descent flows are shown recently to be more accurate and converges faster than the surface formulation, when finite element methods are used for discretizations [16]. Galerkin finite element methods are widely used to discretize partial differential equations defined on arbitrary domains for shape optimization (see, e.g., [3, 22]). For finite element approximations to shape gradients, the numerical accuracy is essential for implementation of optimization algorithms [6]. In [27], numerical comparisons show that the distributed H^1 shape gradient algorithm is more robust and efficient than the usual surface type. Recent convergence analysis is presented for finite element approximations to shape gradients of distributed and surface Eulerian derivatives in eigenvalue optimization [30]. Moreover, we perform convergence analysis for H^1 shape gradients [29] in eigenvalue optimization. The comparisons show the distributed formulation in H^1 shape gradient flows is more efficient in some circumstances.

This paper continues our recent work [28], where we presented convergence analysis for mixed finite element approximations of shape gradients in L^2 flows associated with the Stokes equation. Theoretical as well as numerical evidences therein show that the distributed shape gradient contained in the expression of Eulerian derivative is more accurate and converges faster than the surface shape gradient. In [28], however, neither theoretical analysis nor numerical evidence was provided to show that the numerical H^1 shape gradient descent algorithm based on the distributed formulation is more effective or advantageous than that based on the surface shape gradient for shape optimization. We recently presented numerical evidence that the H^1 shape gradient flow using distributed formulation is more effective for shape reconstruction [18]. But no numerical evidence or theoretical analysis has shown that the distributed shape gradient algorithm is more effective for minimization of drag or energy dissipation.

In this paper, we investigate the performance of H^1 shape gradient flows and offer possible theoretical explainations and numerical evidence for showing advantages of distributed H^1 shape gradient flows in optimal shape design. The models we consider for minimization of energy dissipation or drag in fluids have many applications in industry, e.g., aerodynamics [19] and hemodynamics [21]. We consider mixed finite element approximations to the Stokes equation and finite element approximations to the H^1 gradient flows associated with both boundary and volume formulations of Eulerian derivatives. For the L^2 shape gradient flow, we first generalize the convergence result $\mathcal{O}(h^2)$ of [28] for MINI element to $\mathcal{O}(h^{2s})$ $(0 < s \leq 1)$ under more general and (possible) less regularity assumption on the domain and solution.