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NUMERICAL ANALYSIS OF A NONLINEAR SINGULARLY PERTURBED DELAY VOLTERRA INTEGRO-DIFFERENTIAL EQUATION ON AN ADAPTIVE GRID*

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Abstract

In this paper, we study a nonlinear first-order singularly perturbed Volterra integro-differential equation with delay. This equation is discretized by the backward Euler for differential part and the composite numerical quadrature formula for integral part for which both an a priori and an a posteriori error analysis in the maximum norm are derived. Based on the a priori error bound and mesh equidistribution principle, we prove that there exists a mesh gives optimal first order convergence which is robust with respect to the perturbation parameter. The a posteriori error bound is used to choose a suitable monitor function and design a corresponding adaptive grid generation algorithm. Furthermore, we extend our presented adaptive grid algorithm to a class of second-order nonlinear singularly perturbed delay differential equations. Numerical results are provided to demonstrate the effectiveness of our presented monitor function. Meanwhile, it is shown that the standard arc-length monitor function is unsuitable for this type of singularly perturbed delay differential equations with a turning point.

Mathematics subject classification: 65L05, 65L20, 65L50
Key words: Delay Volterra integro-differential equation, Singularly perturbed, Error analysis, Monitor function

1. Introduction

In this paper, we consider the following nonlinear singularly perturbed delay Volterra integro-differential equation in the interval $\bar{I} = [0, T]$:

$$Lu(t) := \varepsilon u'(t) + a(t)u(t) + \int_0^t f(s, u(s), u(s-r))ds = 0, \qquad t \in I, \tag{1.1}$$

$$u(t) = \varphi(t), \ t \in I_0, \tag{1.2}$$

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where $0 < \varepsilon \ll 1$ is a perturbation parameter, $I = (0, T]$, $I_0 = [-r, 0]$ and $a(t)$, $\varphi(t)$ are given sufficiently smooth functions. We assume that $f(t, u, v) \in C^1(\bar{I} \times \mathbb{R} \times \mathbb{R})$, and there exist three positive constants b^*, c^* and α, such that

$$\left|\frac{\partial f}{\partial u}\right| \le b^*, \quad \left|\frac{\partial f}{\partial v}\right| \le c^*, \quad a(t) \ge \alpha > 0. \tag{1.3}$$

It is well known that delay Volterra integro-differential equations (DVIDE's) arise widely in scientific fields such as physics, medicine, biology, ecology and so on [1–4]. These equations pay an important role in natural science and modeling diverse problems of engineering, and there has been tremendous interest in developing numerical approaches for DVIDE's (see [5, 6] and the references therein). When the highest order derivative term of this kind of problems is multiplied by a small parameter ε, these are said to be singularly perturbed delay Volterra integro-differential equations (SPDVIDE's). As $\varepsilon \to 0$, the exact solution of these equations may have a boundary layer(s) or interior layer(s). Therefore, it is very important to design a suitable numerical method for these problems.

As we know, the commonly used techniques to solve singularly perturbed problems are to construct suitable meshes that are very fine in layer regions. These meshes can be divided into two classes. If the bounds of exact solution and its derivatives are available, one can construct a class of special meshes, see for example the Bakhvalov and Shishkin meshes [7–9]. If the priori information of the exact solution is hard to be obtained, it is desirable to design an a posteriori mesh algorithm, which starts from an initial unsophisticated mesh and then detects the boundary layers and generates an adaptive grid using only the current numerical solution and mesh sizes (see, e.g., [10, 11]).

Over the last decades, there has been a growing interest in the numerical methods for singularly perturbed Volterra integro-differential equations. For example, the authors in [12–15] developed a fitted finite difference scheme on a uniform mesh and gave the convergence results based on the priori information of the exact solution. Şevgin [16] presented a finite difference scheme on a Shishkin mesh and proved that the scheme was first-order convergent in the discrete maximum norm, independently of the perturbation parameter. Huang et.al., [17] proposed an adaptive grid method based on an a posteriori error estimation to solve a singularly perturbed Volterra integro-differential equation with a weakly singular kernel. To the best of our knowledge, only few researchers discussed the numerical methods for SPDVIDE's. For example, Wu and Gan [18] studied the linear multistep method for SPDVIDE's. The authors in [19] constructed a finite difference scheme on a Shishkin mesh for a linear first-order SPDVIDE's. Yapman et.al., [20] considered a quasilinear SPDVIDE and proposed a fitted difference scheme on a uniform mesh, which was first-order uniform convergence in the perturbation parameter. It should be pointed out that the existing methods to solve singularly perturbed delay differential equations need the priori information of the exact solution (see, e.g., [21]). Recently, Das [22] developed an adaptive grid method based on an a posteriori error estimation to solve a system of nonlinear singularly perturbed delay initial value problems, but he didn't give the convergence analysis for the priori error estimation and the assumption of the a posteriori error estimation was too strict.

Motivated by literature [22], this paper will develop an adaptive grid method for problem (1.1)-(1.2). At first, in order to avoid solving a system of nonlinear equations, we utilize the backward Euler formula to discrete the first-order derivative and left rectangle formula to approximate the integral term of problem (1.1)-(1.2). Then, based on the stability result and