Journal of Computational Mathematics
Vol.40, No.3, 2022, 335–353.

http://www.global-sci.org/jcm
doi:10.4208/jcm.2010-m2020-0132

DIRECT IMPLEMENTATION OF TIKHONOV REGULARIZATION FOR THE FIRST KIND INTEGRAL EQUATION*

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Abstract

A common way to handle the Tikhonov regularization method for the first kind Fredholm integral equations, is first to discretize and then to work with the final linear system. This unavoidably inflicts discretization errors which may lead to disastrous results, especially when a quadrature rule is used. We propose to regularize directly the integral equation resulting in a continuous Tikhonov problem. The Tikhonov problem is reduced to a simple least squares problem by applying the Golub-Kahan bidiagonalization (GKB) directly to the integral operator. The regularization parameter and the iteration index are determined by the discrepancy principle approach. Moreover, we study the discrete version of the proposed method resulted from numerical evaluating the needed integrals. Focusing on the nodal values of the solution results in a weighted version of GKB-Tikhonov method for linear systems arisen from the Nyström discretization. Finally, we use numerical experiments on a few test problems to illustrate the performance of our algorithms.

Mathematics subject classification: 45A05, 45Q05, 45N05, 45P05, 65F22, 65F10, 65R32.
Key words: First kind integral equation, Golub-Kahan bidiagonalization, Tikhonov regularization, Quadrature Discretization.

1. Introduction

The first kind integral equations are an important tool to describe the relationship between hidden information and available noisy data. This situation occurs in many applications such as medical imaging, geophysical prospecting, image deblurring and deconvolution of a measurement instruments response [2, 3, 11, 15, 22, 25, 29]. The general form of a first kind Fredholm integral equation is

$$\int_\Omega k(t,y)\phi(y)dy = f(t), \qquad t \in \Omega, \tag{1.1}$$

where Ω is a Jordan measurable and closed bounded set in $\mathbb{R}^m$, for some $m \geq 1$ [1, 21]. The function ϕ is unknown and the kernel k and the right-hand side f are given. By defining the operator $K : L^2(\Omega) \to L^2(\Omega)$

$$(K\phi)(t) := \int_\Omega k(t,y)\phi(y)dy, \qquad t \in \Omega,$$

* Received May 25, 2020 / Revised version received August 18, 2020 / Accepted October 19, 2020 / Published online September 3, 2021 /
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the operator form of (1.1) can be written as

$$K\phi = f. \tag{1.2}$$

When k is continuous, the operator K is compact and so K^{-1} is not continuous, i.e., the solution ϕ does not depend continuously on the data f and is sensitive to small changes in f. This is important from the point that in all applications the data will be measured quantities and the right-hand side f is of the form

$$f = f^{exact} + e, \tag{1.3}$$

where e and f^{exact} denote the perturbation and the unknown error-free right-hand side, respectively. On this basis, the solution obtained by numerical methods proposed to the second kind Fredholm integral equations, can be very far from the solution of unperturbed problem

$$K\phi^{exact} = f^{exact}. \tag{1.4}$$

Hence to compute less sensitive approximations of ϕ^{exact}, some regularization must be employed, i.e., Eq. (1.2) must be replaced by a nearby equation having better numerical properties [12,20]. We will assume that a fairly accurate of the norm of e in (1.3),

$$\|e\|_{L^2} = \delta, \tag{1.5}$$

is known.

The most popular regularization method was introduced by Tikhonov [28] consisting in solving the problem

$$\min_{\phi\in L^2(\Omega)} \{\|K\phi - f\|_{L^2}^2 + \alpha\|\phi\|_{L^2}^2\}, \tag{1.6}$$

where $\alpha > 0$ is called the regularization parameter. Since K is bounded, the minimization problem (1.6) has a unique solution ϕ_α which is given by the solution of the equation

$$\alpha\phi_\alpha + K^*K\phi_\alpha = K^*f, \tag{1.7}$$

where K^* denotes the adjoint of the operator K [20, 21].

Based on the available error level (1.5), a suitable value of α can be determined by the famous method discrepancy principle; that is, the parameter $\alpha(\delta)$ be chosen so that

$$\|K\phi_\alpha - f\| = \eta\delta,$$

where $\eta \geq 1$ is a user-specified constant independent of δ. From [20, Section 2.5], the discrepancy principle based on the Tikhonov regularization method is admissible, i.e., $\alpha(\delta) \to 0$ and

$$\phi_{\alpha(\delta)} \to 0, \qquad \delta \to 0.$$

A difficulty with continuous problems such as (1.1) is that they cannot be represented and manipulated on a computer. A general approach is to discretize the original problem (1.1) and regularize the final linear system of equations

$$Ax = b, \qquad A \in \mathbb{R}^{N\times N}, \qquad x, b \in \mathbb{R}^N, \tag{1.8}$$

which is known as a discrete inverse problem [12]. As a result, the solution ϕ is approximated in a finite number of points in the domain. For example, when the Nyström method is used, the