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APPLICATION OF THE FACTORIZATION METHOD TO RECOVER CUTS WITH OBLIQUE DERIVATIVE BOUNDARY CONDITION*

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Abstract

Direct and inverse problems for the scattering of cracks with mixed oblique derivative boundary conditions from the incident plane wave are considered, which describe the scattering phenomenons such as the scattering of tidal waves by spits or reefs. The solvability of the direct scattering problem is proven by using the boundary integral equation method. In order to show the equivalent boundary integral system is Fredholm of index zero, some relationships concerning the tangential potential operator is used. Due to the mixed oblique derivative boundary conditions, we cannot employ the factorization method in a usual manner to reconstruct the cracks. An alternative technique is used in the theoretical analysis such that the far field operator can be factorized in an appropriate form and fulfills the range identity theorem. Finally, we present some numerical examples to demonstrate the feasibility and effectiveness of the factorization method.

Mathematics subject classification: 35R30, 35P25, 78A46.
Key words: Direct and inverse scattering, Oblique derivative, Crack, The factorization method.

1. Introduction

Scattering of acoustic, electromagnetic and elastic waves by arcs or screens usually leads to boundary value problems in the exterior of open curves with Dirichlet, Neumann or impedance boundary condition. In this paper, we consider the diffraction of tidal waves by spits or reefs. Assume that one surfaces of the cracks are affected by daily rotation of the Earth, which leads to the mixed oblique derivative boundary condition on the surfaces of the cracks. See [22] for the derivation of the oblique derivative boundary condition for Helmholtz equation. To be precise, let the cross section of the spit or reef be an open smooth curve $\Sigma \subset \mathbb{R}^2$ without self-intersections. We assume that Σ can be extended to an arbitrary piecewise smooth, simply connected and closed curve ∂D, enclosing a bounded domain $D \subset \mathbb{R}^2$. The unit outward normal vector and the corresponding tangent vector on the boundary ∂D is denoted by ν and

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τ, respectively. The direction of τ is chosen such that it will coincide with the direction of ν if ν is rotated anticlockwise through an angle of $\frac{\pi}{2}$. Let $u^i = e^{ikx\cdot d}, d \in \Omega$ (the unit sphere of $\mathbb{R}^2$) be the time-harmonic incident plane wave and the corresponding scattered field is denoted by u^s, which can be modeled by the Helmholtz equation

$$\Delta u^s + k^2 u^s = 0 \quad \text{in } \mathbb{R}^2 \backslash \overline{\Sigma}, \tag{1.1}$$

where

$$k^2 := \frac{\omega^2 - p_0^2}{g_0 h_0} > 0$$

with ω being the frequency of the time-harmonic motion, p_0 being the Coriolis parameter, g_0 being the acceleration due to gravity and h_0 being the depth of the ocean. The total field $u := u^i + u^s$ satisfies the mixed boundary conditions of the form

$$\frac{\partial u_-}{\partial \nu} + i\lambda \frac{\partial u_-}{\partial \tau} = 0 \quad \text{on } \Sigma, \tag{1.2}$$

$$u_+ = 0 \qquad\qquad \text{on } \Sigma, \tag{1.3}$$

where the real and constant parameter $\lambda := \frac{p_0}{\omega}$ satisfies $|\lambda| < 1$. The notation $\frac{\partial u_-}{\partial \nu}$ means $\lim_{q\to 0^+} \nu \cdot \nabla u(x - q\nu)$, $\frac{\partial u_-}{\partial \tau}$ represents $\lim_{q\to 0^+} \tau \cdot \nabla u(x - q\tau)$ and $u_+ = \lim_{q\to 0^+} u(x + q\nu)$ for $x \in \Sigma$. The limits here should be understood in terms of uniform convergence. Furthermore, the scattered field u^s satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} \sqrt{r} \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0, \quad r = |x|, \tag{1.4}$$

which holds uniformly with respect to $\hat{x} = x/|x| \in \Omega$. It is known that u^s has the following asymptotic representation

$$u^s(x, d) = \frac{e^{ik|x|}}{\sqrt{|x|}} \left\{ u^\infty(\hat{x}, d) + O\left(\frac{1}{|x|} \right) \right\} \quad \text{as} \quad |x| \to \infty \tag{1.5}$$

uniformly for all directions $\hat{x}$, here u^∞ is the far field pattern of the scattered field u^s. The detailed derivation of this mathematical formulation is given in [14].

The very similar direct scattering problem is studied in [18] by using the integral equation method via a sum of angular and single-layer potentials. The problem is reduced to a Cauchy singular integral equation, which is an uniquely solvable Fredholm equation of the second kind in the classical function space. In this paper, we will use a combined single-layer potential, double-layer potential and the tangential potential to solve the direct scattering problem (1.1)-(1.4) in Sobolev space. Due to some relationships between those potentials and the properties of the related boundary integral operators, the obtained boundary integral system is proved to be a Fredholm equation. The result is shown to coincide with the classical results for the usual scattering problems from crack, see [3, 4, 10] for examples. For the direct scattering problem by obstacle with oblique derivative boundary condition, Martin [22] use only single-layer potential to deduce a quasi-Fredholm integral equation which involves a Cauchy singular integral, and recently, Wang and Liu [29] show the solvability of the direct scattering problem by the Lax-Phillips method. For more articles related to derivative boundary value problems, we recommend the literatures [6, 19, 26, 30, 31].

The inverse problem (**IP**) we are interested in is to reconstruct the crack Σ from the knowledge of the far field pattern $u^\infty(\hat{x}, d)$ for all incident directions d, observation directions $\hat{x}$ at