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INVERSION OF TRACE FORMULAS FOR A STURM-LIOUVILLE OPERATOR*

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Abstract

This paper revisits the classical problem "Can we hear the density of a string?", which can be formulated as an inverse spectral problem for a Sturm-Liouville operator. Instead of inverting the map from density to spectral data directly, we propose a novel method to reconstruct the density based on inverting a sequence of trace formulas which bridge the density and its spectral data clearly in terms of a series of nonlinear integral equations. Numerical experiments are presented to verify the validity and effectiveness of the proposed numerical algorithm. The impact of different parameters involved in the algorithm is also discussed.

Mathematics subject classification: 65F18, 34B24.
Key words: Inverse spectral problem, Sturm-Liouville operator, Trace formulas.

1. Introduction

Studies on inverse spectral problems have been intensive, ranging from mathematical theory to engineering applications. Besides their own interests, inverse spectral problems also have close connections with inverse scattering theory (see, e.g., [1,2]) and inverse boundary value problems (cf. [5,7] and the references therein). There is vast literature of research on this subject, see the survey [3] and the references therein. In this paper, we will propose a novel numerical scheme for an inverse spectral problem related to the classical Sturm-Liouville eigenvalue problem:

$$\begin{aligned} &-\frac{\mathrm{d}^2u}{\mathrm{d}x^2}=\lambda\rho u, \qquad && x\in[0,1],\\ &u(0)=u(1)=0, \end{aligned} \tag{1.1}$$

where $\rho\in L^\infty(0,1)$ and ρ is positive. It models a vibrating string with density ρ. Denote $\{\lambda_k(\rho)\}_{k=1}^\infty$ to be the eigenvalues for the above problem. The *inverse spectral problem* is to recover the density ρ from those eigenvalues $\{\lambda_k(\rho)\}_{k=1}^\infty$. It is well-known that to avoid nonuniqueness, we need to assume $\rho(x)$ is even with respect to $x=\frac{1}{2}$, that is $\rho(x)=\rho(1-x)$.

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If $\rho \in C^2([0,1])$, then the *Liouville transformation*

$$\sigma(x) = \sqrt{\rho(x)}, \quad f(x) = \rho(x)^{1/4}, \quad L = \int_0^1 \sigma(s)\mathrm{d}s,$$
$$t(x) = \frac{1}{L}\int_0^x \sigma(s)\mathrm{d}s, \quad v(t) = f(x(t))u(x(t))$$

reduces the inverse problem to the problem of recovering $q(t)$ in

$$\begin{aligned} &-\frac{\mathrm{d}^2 v}{\mathrm{d}t^2} + qv = L^2\lambda v, \qquad t \in [0,1], \\ &v(0) = v(1) = 0, \end{aligned} \tag{1.2}$$

from the eigenvalues $\{L^2 \cdot \lambda_k(\rho)\}_{k=1}^{\infty}$, where

$$q(t) = L^2 \frac{f(x)}{\rho(x)} \left(\frac{f'(x)}{f(x)^2}\right)' \Bigg|_{x=x(t)}. \tag{1.3}$$

The real-valued number L can be recovered from spectral data [11]. And the uniqueness of q in (1.2) is guaranteed if q is assumed to be symmetric about $x = \frac{1}{2}$. For more details, we refer to [8] and the discussions therein. It is still notable that the problems for (1.1) and (1.2) might be slightly different. First, the recovery of $\rho(x)$ from $q(x)$ from the relation (1.3) might need some additional information. Second, we need some regularity assumption on $\rho(x)$ to do the *Liouville transformation.* In this paper, we will directly deal with problem (1.1), and hence only need to assume $\rho(x)$ is bounded, positive and even with respect to $x = \frac{1}{2}$ for the discussions below.

Numerical methods for one dimensional inverse Sturm-Liouville problem (1.1) and (1.2) abound. We refer to [10–12] and the references therein. However, most methods rely on the one-dimensional nature of the problem itself. In this paper, we give a numerical method for the reconstruction of ρ in (1.1) based on trace formulas. We believe that this method can be generalized to some other inverse spectral problems as long as trace formulas are available.

To solve inverse problems, if we intend to utilize a gradient-based iterative algorithm, we need its Fréchet differentiability of the forward map. However, generally speaking, the differentiability of the map

$$\rho \to \{\lambda_k(\rho)\}_{k=1}^{\infty}, \tag{1.4}$$

is a very delicate issue. Perturbation theory of eigenvalues and eigenfunctions has been studied by Kato [6], but is quite inaccessible. Although this is not a problem for the inverse spectral problem related to (1.1) due to the fact that all eigenvalues are simple and well-ordered, and many methods do not directly invert this map because of other concerns, one needs to bear in mind that it might be a serious issue for some higher dimensional or non-Hermitian problems. In this article, a novel inversion scheme is proposed based on a different map, arising from trace formulas, of which the Fréchet differentiability is *well guaranteed.* For (1.1), the new map is given as follows

$$\rho \to \left\{\sum_{k=1}^{\infty} \lambda_k^{-s}\right\}_{s=1}^{\infty}. \tag{1.5}$$

To be more precise, we will actually invert the following map, for the sake of numerical stability,

$$\rho \to \left\{\sum_{k=1}^{\infty} \mathcal{P}_n(\lambda_k^{-1})\right\}_{n=1}^{\infty}, \tag{1.6}$$