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NUMERICAL ANALYSIS OF A PROBLEM INVOLVING A VISCOELASTIC BODY WITH DOUBLE POROSITY*

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Abstract

We study from a numerical point of view a multidimensional problem involving a viscoelastic body with two porous structures. The mechanical problem leads to a linear system of three coupled hyperbolic partial differential equations. Its corresponding variational formulation gives rise to three coupled parabolic linear equations. An existence and uniqueness result, and an energy decay property, are recalled. Then, fully discrete approximations are introduced using the finite element method and the implicit Euler scheme. A discrete stability property and a priori error estimates are proved, from which the linear convergence of the algorithm is derived under suitable additional regularity conditions. Finally, some numerical simulations are performed in one and two dimensions to show the accuracy of the approximation and the behaviour of the solution.

Mathematics subject classification: 65M60, 37N15, 74F05, 65M12.
Key words: Viscoelasticity with double porosity, Finite elements, A priori estimates, Numerical simulations.

1. Introduction

Porous materials are present in the real life. Two easy examples could be a fairy cake or a piece of limestone travertine. Things which are part of our daily life. But we can also think in some other more sophisticated materials which are currently considered to repair injuries in bones. At the same time, we can also consider porous materials with cracks or fissures in their skeletons. In this situation, we have two different structures. It is worth considering a mathematical theory to describe such materials with porous structure at two levels. In that case, it is usual to speak about macro-porosity and micro-porosity.

Thermoelastic structures with two porosities have been considered in the literature. This problem have attracted much attention in the last twenty years. First models to describe them were based on the Darcy law and the unknowns of the model correspond to the displacement and the pressures at the macro and micro porosities.

It is known that the easiest extension of the classical theory of elasticity is the one proposed by Nunziato and Cowin [1], where the behaviour of elastic solids with voids is considered. The model looks the material with an elastic skeleton and the interstices are voids of the material.

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The number of contributions for such materials is huge. We here recall only a few examples [2–16]. This theory is totally accepted and it is currently used to describe geological materials such as soils or rocks, but also manufactured materials as ceramics and pressed powders. Applications to biomedicine are also relevant.

In 2014 Ieşan and Quintanilla considered the extension of elastic solids with voids to model elastic materials with double porosity. In this case, we have three levels of structures: the elastic skeleton and the two porous components. All the structures have consequences into the others. We cannot modify the state of one of them without consequences into the two others. Much research has been considered recently to understand these materials [16–28]. We can say that this model has been successful and many people is working on it. It is worth saying that, in several recent contributions, the authors provide explicit values for the constitutive parameters (see among others, [25, 29–33]).

In this contribution, we assume that the skeleton of the material is linearly viscoelastic and that two viscous effects are imposed on the porous structures. In the recently published paper [19], the authors proved the analyticity of the solutions whenever the internal energy and the dissipation were assumed strongly positive. Semigroup arguments were used. Two relevant consequences of the analyticity in this case are the exponential stability of the solutions and the impossibility of localization for the solutions. In the present work, we continue the analysis of this model, but from a numerical point of view now. In this sense, we provide a fully discrete approximation by using the classical finite element method and the implicit Euler scheme. A discrete stability property and a priori error estimates are obtained. Numerical simulations in dimension one and two are performed to clarify the accuracy of the algorithm and the behaviour of the material.

The paper is outlined as follows. The mathematical model is described in Section 2 following [19], deriving its variational formulation. An existence and uniqueness result, and an energy decay property, proved in [19], are also stated. Then, in Section 3 a numerical scheme is introduced, based on the finite element method to approximate the spatial domain and the forward Euler scheme to discretize the time derivatives. A discrete stability property is proved, a priori error estimates are deduced for the approximative solutions and, under suitable regularity assumptions, the linear convergence of the algorithm is obtained. Finally, some one- and two-dimensional numerical simulations are presented in Section 4.

2. Mathematical Model

In this section, we present a brief description of the model (details can be found in [19]). We deal with a viscoelastic body with double porosity, restricting ourselves, for the sake of simplicity, to the case of isotropic and homogeneous materials with centre of symmetry.

Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, be the domain and denote by $[0, T]$, $T > 0$, the time interval of interest. The boundary of the body $\Gamma = \partial\Omega$ is assumed to be Lipschitz, with outward unit normal vector $\boldsymbol{\nu} = (\nu_i)_{i=1}^d$. In order to simplify the writing, we do not indicate the dependence of the functions on point $\boldsymbol{x} = (x_j)_{j=1}^d \in \Omega$ and time $t \in [0, T]$. As usual, a subscript after a comma represents its spatial derivative with respect to the prescribed variable, i.e. $f_{i,j} = \partial f_i / \partial x_j$. The time derivatives are represented by a point for the first order, and two points for the second order, over each variable. The repeated index notation is used for the summation.

We denote by $\boldsymbol{u}$, φ and ψ the displacement field, the volume fraction field corresponding to pores and the volume fraction field corresponding to fissures, respectively.