STRONG CONVERGENCE OF THE EULER-MARUYAMA METHOD FOR NONLINEAR STOCHASTIC VOLTERRA INTEGRAL EQUATIONS WITH TIME-DEPENDENT DELAY*

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Abstract

We consider a nonlinear stochastic Volterra integral equation with time-dependent delay and the corresponding Euler-Maruyama method in this paper. Strong convergence rate (at fixed point) of the corresponding Euler-Maruyama method is obtained when coefficients f and g both satisfy local Lipschitz and linear growth conditions. An example is provided to interpret our conclusions. Our result generalizes and improves the conclusion in [J. Gao, H. Liang, S. Ma, Strong convergence of the semi-implicit Euler method for nonlinear stochastic Volterra integral equations with constant delay, Appl. Math. Comput., 348(2019)385-398.]

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1. Introduction and Main Result

Stochastic Volterra-type integral equations become more and more important since there are various applications in many fields such as physics, medical sciences, engineering, finance and so on. Strong existence and uniqueness of the nonlinear Volterra-type stochastic integral equations were discussed in [2] by Ito in 1979. However, the stochastic Volterra-type equations can not be solved explicitly for most cases. So numerical methods have been playing important roles in studying these equations.

Recently convergence of different numerical methods for nonlinear stochastic Volterra integral equations has attracted more and more attention. For example, [9] considered the convergence of a numerical technique based on a combination of the Picard iteration method and hat basis functions under global Lipschitz conditions, [10] considered Euler schemes for stochastic Volterra equations with singular kernels (the coefficients are both Lipschitz continuous with space variable), [8] proposed collocation technique based on delta function approximations for nonlinear stochastic Itô-Volterra equations, [7] investigated strong superconvergence of the Euler-Maruyama method for linear stochastic Volterra integral equations, and then [1] generalized [7] to the semi-implicit Euler methods of following nonlinear Volterra integral equations

$$X(t) = \phi(t) + \int_{t-\tau}^{t} \sigma_1(t-s) f(X(s)) ds + \int_{t-\tau}^{t} \sigma_2(t-s) g(X(s)) dW(s), \quad t \in [0, T]$$

with global Lipschitz condition and obtained that the convergence order of this numerical solution is 0.5.

As far as we know, few literatures have been considered for the convergence of numerical methods for stochastic Voterra integral equations under the local Lipschitz condition. On the other hand, although many systems depend on the past states of their own, the delay might not be constant. For example, in the pandemic of COVID-19, when they are infected, many people will have the symptoms after different period of time, then the number of people with symptoms must be a system with time-dependent delay, so the systems with time-dependent delays become more and more important.

In this work, we are concerned with the following nonlinear stochastic Voterra integral equation with time-dependent delay under local Lipschitz and linear growth conditions of f and g:

$$X(t) = \phi(t) + \int_{t-\delta(t)}^t \sigma_1(t-s)f(X(s))ds + \int_{t-\delta(t)}^t \sigma_2(t-s)g(X(s))dW(s), \quad t \in [0,T] \quad (1.1)$$

with the initial value

$$X_0 = \xi = \{\xi(\theta), \theta \in [-\tau, 0]\} \in C^b_{\mathscr{F}_0}([-\tau, 0]; \mathbb{R}^d).$$

Here $C^b_{\mathscr{F}_0}([-\tau, 0]; \mathbb{R}^d)$ denotes the set of measurable function $\xi : (t, \omega) \to \xi(t, \omega)$ such that for any fixed $\omega \in \Omega$, $\xi(\cdot, \omega) \in C([-\tau, 0]; \mathbb{R}^d)$, and for any fixed $t \in [-\tau, 0]$, $\xi(t, \cdot)$ is a bounded $\mathscr{F}_0$ measurable $\mathbb{R}^d$ valued random variable. $W(t)$ is m-dimensional standard Brownian motion, $\delta(t) \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ such that $\delta(0) = \tau, \delta'(t) < 1$. Here and from now on, $|x| = (\sum_{i=0}^d x_i^2)^{\frac{1}{2}}$. Moreover, for any fixed t, $\phi(t)$ is a random variable, $\sigma_1, \sigma_2 : \mathbb{R}_+ \to \mathbb{R}$, and f and g are $\mathbb{R}^d$-valued and $d \times m$-matrix valued measurable functions, respectively.

Assumption 1.1. *For any fixed $R > 0$, there is $L_R > 0$ such that for all $|x| \bigvee |y| \leq R$,*

$$|f(x) - f(y)| \vee \|g(x) - g(y)\| \leq L_R|x - y|. \quad (1.2)$$

Here $\|\cdot\|$ denotes the trace norm of a matrix.

As interpreted in [5], Remark 2.1, we can always select sufficiently small $\Delta^* > 0$ and a strictly monotone decreasing function $h : (0, \Delta^*] \to (0, \infty)$ such that

$$\lim_{\Delta \to 0} h(\Delta) = \infty, \quad \lim_{\Delta \to 0} L^2_{h(\Delta)}\Delta = 0. \quad (1.3)$$

Assumption 1.2. *There is a constant $K > 0$ such that for all $x \in \mathbb{R}^d$,*

$$|f(x)| \vee \|g(x)\| \leq K(1 + |x|). \quad (1.4)$$

To approximate the exact solution of Eq. (1.1), we take Δ as the stepsize of the numerical solution satisfying $\Delta = \frac{T}{N} = \frac{\tau}{n_0}$. Let $t_n = n\Delta, n = -n_0, -n_0 + 1, \cdots, 1, \cdots, N - 1, t_N = T$.

The Euler-Maruyama method for (1.1) is defined as follows:

$$X_n = \phi(t_n) + \sum_{l=n-n_\Delta}^{n-1} \int_{t_l}^{t_{l+1}} \sigma_1(t_n - s)ds \cdot f(X_l)$$

$$+ \sum_{l=n-n_\Delta}^{n-1} \sigma_2(t_n - t_l)g(X_l)\Delta W_l, \quad n \geq 1, \quad (1.5a)$$

$$X_n = \xi(t_n), \quad n = -n_0, -n_0 + 1, \cdots, -1, 0 \quad (1.5b)$$