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# A NEW HYBRIDIZED MIXED WEAK GALERKIN METHOD FOR SECOND-ORDER ELLIPTIC PROBLEMS*

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**Abstract**

In this paper, a new hybridized mixed formulation of weak Galerkin method is studied for a second order elliptic problem. This method is designed by approximate some operators with discontinuous piecewise polynomials in a shape regular finite element partition. Some discrete inequalities are presented on discontinuous spaces and optimal order error estimations are established. Some numerical results are reported to show super convergence and confirm the theory of the mixed weak Galerkin method.

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*Key words:* Weak Galerkin, Weak gradient, Hybridized mixed finite element method, Second order elliptic problems.

## 1. Introduction

Hybridized mixed finite element method for second order elliptic boundary value problems on polygonal domains provides approximations of the solution in terms of elementwise given scalar and vector valued functions and a multiplier on the set of interior edges or faces of the underlying triangulation of the domain. Recently, there is an attracted particular interest within a unified framework for hybridization of mixed and discontinuous Galerkin methods (see, e.g., [2, 3, 7, 8, 10–12, 14, 16, 17, 19, 20] and the references therein). In the literature, the hybridized mixed Galerkin method possesses local mass conservation and continuity of fluxes. That preserves some mathematical properties of physical systems in many applications such as modeling groundwater flow, reactive transport in porous media. That allowed to make a competitive numerical technique to provide a good approximation for both the velocity and pressure. Hybridized mixed Galerkin method was a clever implementation technique that contains more information than the solution obtained with numerical classical methods, it was

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described in [7]. Lately, a new perspective on hybridized mixed Galerkin method was developed and it gained great popularity for the reasons that it provides very accurate approximations of the unknown and its flux, and it conserves mass locally. In [22] authors developed the hybridized Galerkin finite element method by introducing a weak formulation based on a discrete weak gradient. However, in [21] a mixed weak Galerkin method was introduced and analyzed using a weak divergence. Both methods are employed for second order elliptic problems with Dirichlet conditions.

In this paper, we introduce and analyse a new formulation of weak Galerkin mixed finite element method based on the weak gradient, for solving numerically the second order elliptic boundary value problem. We analyse and we test, numerically, the performance of our new mixed formulation. Our contribution is part of an ongoing effort to develop and test a series of weak Galerkin methods for involving some partial differential equations. Weak Galerkin finite element method is an extension of the local discontinuous Galerkin method studied by authors in [9, 18, 23–26]. Which consists to approximate differential operators in the weak formulation by weak forms as generalized distributions. This method contributes considerably to the enrichment of numerical methods thanks to its flexibility and its inexpensive cost in programming even for non-regular domains. The weak Galerkin method differs from the LDG method by the use of weak derivative and weak functions.

The problem studied for most of this paper is the second order boundary value problem

$$\begin{aligned} \alpha\mathbf{q} + \nabla u = 0 \quad &\text{in} \quad \Omega \subset \mathbb{R}^2, \\ \nabla \cdot \mathbf{q} = f \qquad &\text{in} \quad \Omega \subset \mathbb{R}^2 \end{aligned} \tag{1.1}$$

together with the Dirichlet boundary condition $u = 0$ in $\partial\Omega$. Here $\Omega$ is a bounded polygon in $\mathbb{R}^2$, $\alpha = (\alpha_{i,j}(x))$ is a symmetric and uniformly positive definite $2 \times 2$ matrix and supposed that in $[L^\infty(\Omega)]^{2^2}$.

A weak formulation for the problem (1.1) consists to find $\mathbf{q} \in L^2(\Omega)^2$ and $u \in H_0^1(\Omega)$ such that

$$\begin{aligned} (\alpha\mathbf{q}, \mathbf{v}) + (\nabla u, \mathbf{v}) = 0 \quad &\forall\, \mathbf{v} \in L^2(\Omega)^2, \\ (\mathbf{q}, \nabla\psi) = -(f, \psi) \qquad &\forall\, \psi \in H_0^1(\Omega) \end{aligned} \tag{1.2}$$

with $L^2(\Omega)$ is the standard space of square integrable functions on $\Omega$, $H_0^1(\Omega)$ is the standard Sobolev space with zero trace on $\partial\Omega$ and $(\cdot,\cdot)$ is the stands for the $L^2$- scalar product in $L^2(\Omega)$.

This paper is presented as follows. Section 2 is done to introduce some notations and spaces. In Section 3, we derive a hybridized mixed weak Galerkin based on weak gradient formulation and we study its well posedness. In Section 4, we define the local projection operators and some inequalities that are essential to the study of error analysis. Also in this section, the properties of the bilinear forms are studied and some error estimation results are proved. Finally, some numerical results are presented and they confirm the theoretical convergence estimates.

## 2. Preliminaries and Notations

For a domain $D$ in $\mathbb{R}^2$, we denote by $H^s(D)^d, d = 1, 2$, the Sobolev spaces of real scalar or vector functions with integer or fractional regularity exponent $s \geq 0$, endowed with the usual norm $\|\cdot\|_{s,D}$ and by $L^2(D)^d, d = 1, 2$, the standard space of square integrable scalar or vector functions on $D$, we refer to [1, 13] for more details.