Journal of Computational Mathematics
Vol.40, No.4, 2022, 519–543.
http://www.global-sci.org/jcm
doi:10.4208/jcm.2011-m2019-0205

AN EXPLICIT MULTISTEP SCHEME FOR MEAN-FIELD FORWARD-BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS*

Yabing Sun
School of Mathematics, Shandong University, Jinan, Shandong 250100, China, and College of Science, National University of Defense Technology, Changsha 410073, China
Email: sunybly@163.com
Jie Yang
School of Mathematics and Statistics, Shandong University, Weihai 264209, China
Email: jieyang@sdu.edu.cn
Weidong Zhao[1)]
School of Mathematics and Finance Institute, Shandong University, Jinan 250100, China
Email: wdzhao@sdu.edu.cn
Tao Zhou
LSEC, Institute of Computational Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China
Email: tzhou@lsec.cc.ac.cn

Abstract

This is one of our series works on numerical methods for mean-field forward backward stochastic differential equations (MFBSDEs). In this work, we propose an explicit multistep scheme for MFBSDEs which is easy to implement, and is of high order rate of convergence. Rigorous error estimates of the proposed multistep scheme are presented. Numerical experiments are carried out to show the efficiency and accuracy of the proposed scheme.

Mathematics subject classification: 60H35, 65C20, 60H10.
Key words: Mean-field forward backward stochastic differential equations, Explicit multistep scheme, Error estimates.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered complete probability space with the filtration $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ generated by an m-dimensional standard Brownian motion $W = (W_t)_{0\leq t\leq T}$. We consider the following decoupled mean-field forward backward stochastic differential equations (MFBSDEs) on $(\Omega, \mathcal{F}, \mathbb{F}, P)$:

$$\begin{cases} X_t^{0,X_0} = X_0 + \int_0^t \mathbb{E}\left[b(s, X_s^{0,x_0}, x)\right]\big|_{x=X_s^{0,X_0}} ds + \int_0^t \mathbb{E}\left[\sigma(s, X_s^{0,x_0}, x)\right]\big|_{x=X_s^{0,X_0}} dW_s, \\ Y_t^{0,X_0} = \xi + \int_t^T \mathbb{E}\left[f(s, \Theta_s^{0,x_0}, \theta)\right]\big|_{\theta=\Theta_s^{0,X_0}} ds - \int_t^T Z_s^{0,X_0} dW_s, \end{cases} \tag{1.1}$$

* Received September 16, 2019 / Revised version received October 14, 2020 / Accepted November 17, 2020 / Published online March 12, 2021 /
1) Corresponding author

where $\Theta_s^{t_0,\eta} = (X_s^{t_0,\eta}, Y_s^{t_0,\eta}, Z_s^{t_0,\eta})$ for $\eta = x_0$ or X_0, and $\theta = (x, y, z)$; $X_0 \in \mathcal{F}_0$ is the initial value of the mean-field stochastic differential equation (MSDE), and $\xi = \mathbb{E}[\Phi(X_T^{0,x_0}, \mu)]|_{\mu = X_T^{0,X_0}}$ is the terminal condition of the mean-field backward stochastic differential equation (MBSDE); $b : [0,T]\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0,T]\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}^{d\times m}$ are respectively referred to as the drift and diffusion coefficients of MSDE, and $f : [0,T]\times\mathbb{R}^d\times\mathbb{R}^p\times\mathbb{R}^{p\times m}\times\mathbb{R}^d\times\mathbb{R}^p\times\mathbb{R}^{p\times m} \to \mathbb{R}^p$ is the generator of MBSDE. A solution triple $(X_t^{0,X_0}, Y_t^{0,X_0}, Z_t^{0,X_0})$ of (1.1) is called an L^2-solution if it is $\mathcal{F}_t$-adapted and square integrable. In general, X_0 and x_0 are of different values, and the triple $(X_t^{0,x_0}, Y_t^{0,x_0}, Z_t^{0,x_0})$ is the solution of the MFBSDEs with $X_0 = x_0$.

The above MFBSDEs admit diverse applications in many research areas such as kinetic gas theory [3], economics and finance [14], quantum mechanics [22], stochastic optimal control problems [17,18,24], mean-field games for large population multi-agent systems [1,9,31], stochastic particle systems with mean-field interactions [2,3,22], to name a few.

While there have been many works on numerical methods for BSDEs and FBSDEs, see e.g., [10–13,19–21,23,33,35–37]. However, there are only a few of works on numerical MFBSDEs. The main difficulty is that the solutions of MFBSDEs depend on the distributions of the forward SDEs which makes it a challenge to construct efficient and accurate numerical schemes.

In our previous works [29,30], we have proposed two one-step schemes: an explicit θ-scheme and an explicit second order scheme for decoupled MFBSDEs. In this work, by adopting the backward orthogonal polynomials in [38], we shall propose an explicit multistep scheme for MFBSDEs. The proposed multistep scheme is easy to implement, yet is of high order rate of convergence. Moreover, we present rigorous error estimates, which show that the scheme admits high order convergence rates if the associate MSDEs are solved by high order schemes. To verify our theoretical finding, we present several numerical experiments, in which the Monte Carlo method is used for approximating the expectations involved in MFBSDEs. The numerical results indeed show that the proposed scheme is stable, effective and is of high order rate of convergence.

The rest of this paper is organized as follows. In Section 2, we introduce some preliminaries which include the backward orthogonal polynomials, the Feynman-Kac formula for MFBSDEs, and the mean-field Itô-Taylor expansion. Our multistep scheme and its error estimates are presented in Section 3 and Section 4, respectively. In Section 5, numerical tests are presented to verify the theoretical results. Finally, we give some concluding remarks in Section 6.

2. Preliminaries

This section presents some preliminaries on the backward orthogonal polynomials, the Feynman-Kac formula in the mean-field case, and the mean-field Itô-Taylor expansions.

2.1. Backward orthogonal polynomials

We recall the backward orthogonal polynomials introduced in [38], which shall play an important role in designing and analyzing our multistep scheme.

Definition 2.1 (Backward orthogonal polynomials). *We call a set of polynomials $\{Q_i(s)\}_{i=0}^{L}$ defined on the interval $[0,1]$ the backward orthogonal polynomials, if for each $i = 0, 1, \ldots, L$, it holds that*

$$\int_0^1 Q_i(s)ds = 1, \quad \int_0^1 Q_i(s)s^j ds = 0, \quad 1 \le j \le i.$$