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GENERAL FULL IMPLICIT STRONG TAYLOR APPROXIMATIONS FOR STIFF STOCHASTIC DIFFERENTIAL EQUATIONS*

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Abstract

In this paper, we present the backward stochastic Taylor expansions for a Ito process, including backward Ito-Taylor expansions and backward Stratonovich-Taylor expansions. We construct the general full implicit strong Taylor approximations (including Ito-Taylor and Stratonovich-Taylor schemes) with implicitness in both the deterministic and the stochastic terms for the stiff stochastic differential equations (SSDE) by employing truncations of backward stochastic Taylor expansions. We demonstrate that these schemes will converge strongly with corresponding order $1, 2, 3, \ldots$. Mean-square stability has been investigated for full implicit strong Stratonovich-Taylor scheme with order 2, and it has larger mean-square stability region than the explicit and the semi-implicit strong Stratonovich-Taylor schemes with order 2. We can improve the stability of simulations considerably without too much additional computational effort by using our full implicit schemes. The full implicit strong Taylor schemes allow a larger range of time step sizes than other schemes and are suitable for SSDE with stiffness on both the drift and the diffusion terms. Our numerical experiment show these points.

Mathematics subject classification: 65C30, 60H35.
Key words: Stiff stochastic differential equations, Approximations, Backward stochastic Taylor expansions, Full implicit Taylor methods.

1. Introduction

We are concerned with numerical methods for the solution of a d-dimensional vector Ito stochastic differential equation

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t, \tag{1.1}$$

where $\{W_t, t \in [0,T]\}$ is an m-dimensional Wiener process with components $W_t^1, W_t^2, \cdots, W_t^m$, which are independent standard Wiener processes on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, a is a d-dimensional vector function from $[0,T] \times \mathbb{R}^d$ to $\mathbb{R}^d$ and b is a $d \times m$-matrix function from $[0,T] \times \mathbb{R}^d$ to $\mathbb{R}^{d\times m}$. We interpret (1.1) as a stochastic integral equation

$$X_t = X_0 + \int_0^t a\left(s, X_s\right) ds + \sum_{j=1}^{m} \int_0^t b^j \left(s, X_s\right) dW_s^j \tag{1.2}$$

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with the i-th component being

$$X_t^i = X_0^i + \int_0^t a^i (s, X_s)\, ds + \sum_{j=1}^m \int_0^t b^{i,j} (s, X_s)\, dW_s^j, \quad i = 1, \ldots, d, \tag{1.3}$$

where the stochastic integrals are Ito stochastic integrals. We call $X = \{X_t, t \in [0,T]\}$ an Ito process.

We also can transform (1.1) into the equivalent Stratonovich form

$$dX_t = \underline{a}(t, X_t)dt + b(t, X_t) \circ dW_t \tag{1.4}$$

with

$$\underline{a}^i(t, X) = a^i(t, X) - \frac{1}{2} \sum_{j=1}^d \sum_{k=1}^m b^{j,k}(t, X) \frac{\partial b^{i,k}}{\partial x_j}(t, X), \quad i = 1, \ldots, d, \tag{1.5}$$

and the form of stochastic integral equation is

$$X_t = X_0 + \int_0^t \underline{a}\,(s, X_s)\, ds + \sum_{j=1}^m \int_0^t b^j\,(s, X_s) \circ dW_s^j \tag{1.6}$$

with the i-th component being

$$X_t^i = X_0^i + \int_0^t \underline{a}^i\,(s, X_s)\, ds + \sum_{j=1}^m \int_0^t b^{i,j}\,(s, X_s) \circ dW_s^j, \quad i = 1, \ldots, d, \tag{1.7}$$

where the stochastic integrals are Stratonovich stochastic integrals. For more details, see [3].

The stochastic differential equation can be applied in many different fields. Examples include population dynamics, protein kinetics, genetics, experimental psychology, neuronal activity, investment finance, option pricing, turbulent diffusion, radio-astronomy, helicopter rotor, satellite orbit stability, biological waste treatment, hydrology, air quality, seismology, structural mechanics, fatigue cracking, optical bistability, nematic liquid crystals, blood clotting dynamics, cellular energetics, Josephson tunneling junctions, communications and stochastic annealing. For more examples and details, we refer to [3].

However, explicit solutions of Eqs. (1.1) are rare in practical applications and numerical methods are necessary. The most efficient and widely applicable approach to solving (1.1) is the simulation of sample paths of time discrete approximations, like Euler scheme, Milstein scheme and so on. In this paper, we focus on the time discrete approximations and consider a time discretization $(\tau)_\Delta$ with

$$0 = \tau_0 < \tau_1 < \cdots < \tau_n < \cdots < \tau_N = T \tag{1.8}$$

of a time interval $[0,T]$, which in the simplest equidistant case has step size

$$\Delta = T/N. \tag{1.9}$$

The simplest time discrete approximation is the Euler scheme. For (1.1), it has the form

$$Y_{n+1} = Y_n + a\,(\tau_n, Y_n)\,\Delta + b\,(\tau_n, Y_n)\,\Delta W \tag{1.10}$$

for $n = 0, 1, \ldots, N-1$ with initial value

$$Y_0 = X_0, \tag{1.11}$$

$$\Delta W = W_{\tau_{n+1}} - W_{\tau_n}. \tag{1.12}$$