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STRONG CONVERGENCE OF THE EULER-MARUYAMA METHOD FOR A CLASS OF STOCHASTIC VOLTERRA INTEGRAL EQUATIONS*

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Abstract

In this paper, we consider the Euler-Maruyama method for a class of stochastic Volterra integral equations (SVIEs). It is known that the strong convergence order of the Euler-Maruyama method is $\frac{1}{2}$. However, the strong superconvergence order 1 can be obtained for a class of SVIEs if the kernels $\sigma_i(t,t)=0$ for $i=1$ and 2; otherwise, the strong convergence order is $\frac{1}{2}$. Moreover, the theoretical results are illustrated by some numerical examples.

Mathematics subject classification: 65L20, 65C40.
Key words: Strong convergence, Stochastic Volterra integral equations, Euler-Maruyama method, Lipschitz condition.

1. Introduction

Many real-world phenomena are subject to random environmental effects and stochastic differential equations (SDEs)

$$dY(t)=f\big(Y(t)\big)dt+g\big(Y(t)\big)dw(t) \tag{1.1}$$

can well model this kind of phenomenon. Since it is difficult to get analytical solutions of SDEs, numerical methods are a good choice. Numerical methods for SDEs have been well studied (see, e.g., [5,12] and the references therein).

Stochastic integral equations (SIEs) play an important role in all kinds of application areas including economy, biology, population dynamics and so on. In recent years, the study of stochastic Volterra integral equations (SVIEs) has attracted the attention of many authors (see, e.g., [1,4,7,9,15–17]). For example, Mao [13] studied the stability of the stochastic Volterra integro-differential equations (SVIDEs)

$$dY(t)=f\big(Y(t),t\big)dt+g\left(\int_0^t G(t-s)Y(s)ds,t\right)dw(t). \tag{1.2}$$

Later on, Mao and Riedle [14] extended these results to a more general SVIDEs, namely,

$$dY(t)=\left[f\big(Y(t),t\big)+g\left(\int_0^t G(t-s)Y(s)ds,t\right)\right]dt+h\left(\int_0^t H(t-s)Y(s)ds,t\right)dw(t). \tag{1.3}$$

Recently, Liang et al. [8] considered the Euler-Maruyama method for the following linear SVIEs of Itô-type:

$$Y(t)=\phi(t)+\int_0^t K_1(t,s)Y(s)ds+\int_0^t K_2(t,s)Y(s)dw(s). \tag{1.4}$$

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In 2019, Yang et al. [17] studied the semi-implicit Euler method for the following nonlinear SVIDEs of Itô-type:

$$\frac{dY(t)}{dt} = f\big(Y(t)\big) + \int_0^t \sigma(t,s)Y(s)dw(s). \tag{1.5}$$

In the same year, Gao et al. [2] investigated the strong convergence analysis of the semi-implicit Euler method for the SVIDEs

$$dY(t) = f\left(Y(t), \int_0^t k_1(t,s)Y(s)ds, \int_0^t \sigma_1(t,s)Y(s)dw(s)\right)dt. \tag{1.6}$$

Motivated by SDE (1.1), SVIDE (1.2), SVIDE (1.3), SVIDE (1.5) and SVIDE (1.6), we consider a class of SVIEs in the form

$$\begin{aligned} Y(t) = \phi(t) &+ \int_0^t f\left(Y(z), \int_0^z k_1(z,s)Y(s)ds, \int_0^z \sigma_1(z,s)Y(s)dw(s)\right)dz \\ &+ \int_0^t \sigma_2(t,s)Y(s)dw(s) \end{aligned} \tag{1.7}$$

for $t \in [0,T]$, with initial data $\phi(t) : [0,T] \to \mathbb{R}$, equipped with $\|\phi\|_\infty = \max_{t\in[0,T]} |\phi(t)|$, where $f : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$, the kernels k_1: $D \to \mathbb{R}$ and σ_i: $D \to \mathbb{R}$. Here $D := \{(t,s) : 0 \le s \le t \le T\}$.

SVIE (1.7) can be regarded as the more generalized type of these models. We apply the Euler-Maruyama method to SVIE (1.7). The outline of the paper is as follows. In Section 2, we introduce some notations and hypotheses. The mean square boundedness of the Euler-Maruyama method is given and its strong convergence is shown, respectively in Section 3 and Section 4. In Section 5, some numerical experiments are used to verify the results obtained from the theory.

2. Preliminary

Throughout this paper, let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\ge0},\mathbb{P})$ denote a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge0}$ satisfying the usual conditions (i.e., it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets), and let $\mathbb{E}$ be the expectation corresponding to $\mathbb{P}$. Let $w(t)$ denote a 1-dimensional Brownian motion defined on the probability space. For $a, b \in \mathbb{R}$, we use $a \vee b$ and $a \wedge b$ for $\max\{a,b\}$ and $\min\{a,b\}$, respectively.

We impose the following four assumptions:

(A1) f satisfies the global Lipschitz condition: There is a positive constant K_1 such that

$$|f(x,y,z) - f(\hat{x},\hat{y},\hat{z})|^2 \le K_1\Big(|x-\hat{x}|^2 + |y-\hat{y}|^2 + |z-\hat{z}|^2\Big) \tag{2.1}$$

for $x, y, z, \hat{x}, \hat{y}, \hat{z} \in \mathbb{R}$.

(A2) The kernels k_1, σ_1 and σ_2 satisfy the global Lipschitz condition: There is a positive constant C such that

$$\big|k_1(t,s) - k_1(\hat{t},\hat{s})\big|^2 \vee \big|\sigma_i(t,s) - \sigma_i(\hat{t},\hat{s})\big|^2 \le C\left(|t-\hat{t}|^2 + |s-\hat{s}|^2\right) \tag{2.2}$$

for $i = 1,2$ and $(t,s), (\hat{t},\hat{s}) \in D$.