Journal of Computational Mathematics
Vol.40, No.4, 2022, 624–648.

http://www.global-sci.org/jcm
doi:10.4208/jcm.2101-m2020-0234

A CONFORMING QUADRATIC POLYGONAL ELEMENT AND ITS APPLICATION TO STOKES EQUATIONS*

Xinjiang Chen and Yanqiu Wang[1)]
Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences,
Nanjing Normal University, Nanjing 210023, China
Email: 554166448@qq.com, yqwang@njnu.edu.cn

Abstract

In this paper, we construct an H^1-conforming quadratic finite element on convex polygonal meshes using the generalized barycentric coordinates. The element has optimal approximation rates. Using this quadratic element, two stable discretizations for the Stokes equations are developed, which can be viewed as the extensions of the P_2-P_0 and the Q_2-(discontinuous)P_1 elements, respectively, to polygonal meshes. Numerical results are presented, which support our theoretical claims.

Mathematics subject classification: 65N38, 65N30.
Key words: Quadratic finite element method, Stokes equations, Generalized barycentric coordinates.

1. Introduction

Recently, polygonal/polyhedral meshes have gained much attention in the finite element community partly due to their flexibility when dealing with complicated domains, curved boundaries, local mesh refining/coarsening, and also to their connection with the Voronoi diagram. Various numerical methods have been developed for solving partial differential equations (PDEs) on polygonal/polyhedral meshes, including but not limited to: the mimetic finite difference method (see the recent survey paper [27]), the finite element method [28, 36–39], the finite volume method [25], the virtual element method (see [1] and references therein), the discontinuous Galerkin (DG) method as well as the hybridized DG method [11, 22], the weak Galerkin method [31], and the hybrid high-order method [13]. Here we are interested in the H^1-conforming finite element method based on the generalized barycentric coordinates (GBCs) [17,18,20,43,44]. The GBCs, which are polygonal extensions of the triangular barycentric coordinates, are naturally suitable for constructing first-order accurate H^1-conforming polygonal finite elements. However, because of their non-polynomial nature, constructing second-order accurate, a.k.a. quadratic, GBC-based polygonal elements turns out to be difficult. Rand, Gillette and Bajaj have proposed a serendipity quadratic construction in [32], which uses a minimal amount of degrees of freedom (dofs) only associated with vertices and midpoints of edges. However, their construction is highly technical, and consequently, its implementation, although feasible, becomes impractical especially when the number of vertices in a polygon becomes large. Later, Floater and Lai constructed a more practical serendipity quadratic element in [21] by using combinations of GBCs and triangular local barycentric coordinates. In this paper, We propose a purely GBC-based solution that makes a compromise

* Received September 2, 2020 / Revised version received December 3, 2020 / Accepted January 3, 2021 / Published online February 11, 2022 /
1) Corresponding author

between the number of dofs and the convenience of implementation. Our quadratic element adds at most 3 bubble functions per polygon in addition to the vertex- and edge-based minimal dofs, in exchange for a straight-forward construction and implementation. It is worth pointing out that the bubble structure in our quadratic element can be quite convenient in the stability analysis of the discretization of Stokes equations to be constructed later. Our element is proved to be second-order accurate, i.e., it has interpolation errors of $O(h^2)$ in H^1-seminorm and $O(h^3)$ in L^2-norm, where h is the characteristic mesh size.

The quadratic element can be used to develop second-order discretizations for the Stokes equations on polygonal meshes. Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain. Consider the following Stokes equation:

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f}, & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0, & \text{in } \Omega, \\ \boldsymbol{u} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\boldsymbol{u}$ is the velocity, p is the pressure, and $\boldsymbol{f}$ is the body force. It is well-known that when discretizing Equation (1.1) using the finite element method, the discrete spaces for the velocity and the pressure need to be stable, i.e., satisfying the discrete inf-sup (LBB) condition, in order for the method to converge. Alternatively, one can also use various stabilization techniques [6, 7, 14, 15] to achieve convergence for unstable elements. Readers may refer to the books [4, 24] and references therein for the numerous works in this area.

There are only a few existing works on discretizing the Stokes equations on polygonal meshes using the GBCs. In [40], Talischi et. al. investigated a direct extension of the P_1-P_0/Q_1-P_0 element to polygonal meshes, which according to the same argument in [2] for the mimetic finite difference method, is stable if every internal vertex of the mesh is connected to at most three edges. Later, in [42] Vu-Huu et. al. considered an extension of the P_1-P_1 element, stabilized by a polynomial pressure projection [14]. The same authors of [42] also considered an extension of the MINI element in [41], again with a stabilization to ensure the convergence of the discretization. Recently, Chen and Wang proposed in [8] an extension of the Bernardi-Raugel element [3], which is proved to be stable on general convex polygonal meshes.

All the aforementioned GBC-based discretizations for Stokes equations have only first-order accuracy. Now, using the quadratic polygonal element we have constructed, it is possible to develop second-order discretizations for the Stokes equations on polygonal meshes. Two discretizations will be considered. The first one is a direct extension of the P_2-P_0 element, which is known to be stable on triangular meshes. We call it a G_2-P_0 element, where 'G' means that the element is built upon GBCs. The stability analysis for the G_2-P_0 element is simple. However, the element is known to be 'unbalanced' because of the different approximation rates provided by the discretizations for the velocity and the pressure. Overall it has only $O(h)$ convergence, but can achieve $O(h^2)$ convergence if the exact pressure is zero. The second one is an extension of the Q_2-(discontinuous)P_1 element [35], and will be called a G_2-P_1 element in this paper. Note that a triangular mesh is one type of the polygonal mesh, and it is known that the P_2-(discontinuous)P_1 element is not stable on general triangular meshes [33]. Therefore on triangles, we actually uses the conforming Crouzeix-Raviart element [12] in defining the G_2-P_1 element, that is, the discrete space is enhanced by 1 bubble per triangle for each component of the velocity. We will prove that the G_2-P_1 element is stable and achieves $O(h^2)$ convergence on general convex polygonal meshes.

Throughout the paper, we often use c or C to denote general positive constants independent