Journal of Computational Mathematics
Vol.40, No.4, 2022, 649–666.
http://www.global-sci.org/jcm
doi:10.4208/jcm.2101-m2020-0214

WAVEFORM RELAXATION METHODS FOR LIE-GROUP EQUATIONS*

Yaolin Jiang[1)], Zhen Miao and Yi Lu
School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, China
Emails: yljiang@mail.xjtu.edu.cn, mz91127@126.com, luyi06@gmail.com

Abstract

In this paper, we derive and analyse waveform relaxation (WR) methods for solving differential equations evolving on a Lie-group. We present both continuous-time and discrete-time WR methods and study their convergence properties. In the discrete-time case, the novel methods are constructed by combining WR methods with Runge-Kutta-Munthe-Kaas (RK-MK) methods. The obtained methods have both advantages of WR methods and RK-MK methods, which simplify the computation by decoupling strategy and preserve the numerical solution of Lie-group equations on a manifold. Three numerical experiments are given to illustrate the feasibility of the new WR methods.

Mathematics subject classification: 65L05, 65L06, 65L20.
Key words: Lie-group equations, Waveform relaxation, RK-MK methods, Convergence analysis.

1. Introduction

In this paper, we consider efficient numerical methods for the following ordinary differential equation (ODE) on manifolds ([4,15])

$$Y'(t) = A(t, Y(t))Y(t), \quad t \geq 0, \tag{1.1}$$

with the initial value $Y(0) = Y_0 \in \mathcal{G}$, where $\mathcal{G}$ is a Lie-group, the map $A(t, Y) : [0, +\infty) \times \mathcal{G} \longrightarrow \mathfrak{g}$ is Lipschitz continuous and $\mathfrak{g}$ is the Lie algebra of $\mathcal{G}$. Here $A(t, Y)$ is an $n \times n$ matrix and $A(t, Y)Y$ is the usual matrix product of $A(t, Y) \in \mathfrak{g}$ and $Y \in \mathcal{G}$. It is well known that the solution of (1.1) stays in the Lie-group $\mathcal{G}$ and the equation (1.1) is also called the Lie-group equation.

In order to solve Lie-group equations efficiently, a kind of method called Lie-group method, which is a branch of Geometric Numerical Integration, has received significant attention in recent years. The main advantage of Lie-group methods is that they can preserve the numerical structure of the Lie-group equation on the manifold [2,4–6,10]. The pioneering work of this topic was given by Crouch and Grossman [7], who were the first to introduce numerical methods that evolve on manifolds. They applied a classical Runge-Kutta (RK) method to Lie-group equations by repeatedly freezing and thawing coefficients and keeping the flow in the correct configuration space [3,15,16]. RK methods on Lie-groups have been further studied in [14,17,25–27]. Among them, Munthe-Kaas considered applying the classical RK methods to the Lie algebra equation of Lie-group equations on manifolds and for related work see [8,25–27,31]. Magnus and Fer methods were derived and analysed in [17,31]. These two kinds of algorithms require the

* Received August 17, 2020 / Revised version received November 17, 2020 / Accepted January 4, 2021 / Published online February 16, 2022 /
1) Corresponding author

evaluation of a large number of multivariate integrals which can be computed in a tractable manner by using quadrature schemes and by exploiting the structure of the Lie algebra.

On the other hand, the waveform relaxation (WR) methods come from large scale integrated circuits and have been considered for various applications [18, 19, 21, 23, 30, 32]. They decouple a large system into a number of simplified sub-systems on time-intervals which can be solved independently with their own time integrations. The notable feature of WR methods is decoupling and parallelizing [9, 18, 20]. However, WR methods have not been considered so far for solving Lie-group equations, which motivates this paper.

In this paper, we will present continuous-time and discrete-time WR methods for solving the Lie-group equation (1.1). The discrete-time scheme combines WR methods with RK-MK methods. It preserves the numerical solution of Lie-group equation on a manifold and simplifies the computation in practice. The rest of this paper is organized as follows. In Section 2, some basic knowledge and concepts that we will use in the paper are introduced. In Section 3, the formulation of the novel methods and the convergence are presented. Three numerical experiments are carried out to illustrate the feasibility of the proposed methods in Section 4. Finally, we present some conclusions in the last section.

2. Preliminaries

In this paper, some basic knowledge and concepts will be used and we represent them as follows.

Definition 2.1. *If a differential manifold $\mathcal{G}$ is equipped with a product: $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$ satisfying the following properties:*

- $p \times (q \times r) = (p \times q) \times r$, $\forall p, q, r \in \mathcal{G}$,
- $\exists I \in \mathcal{G}$ *such that* $I \times p = p \times I = p$, $\forall p \in \mathcal{G}$,
- $\forall p \in \mathcal{G}$, $\exists p^{-1} \in \mathcal{G}$, *such that* $p^{-1}p = I$,
- *The maps* $(p, q) \to p \times r$ *and* $p \to p^{-1}$ *are smooth functions,*

  then we call it a Lie-group.

Definition 2.2. *A vector space $\mathfrak{g}$ is called a Lie algebra if it is equipped with a bilinear skew-symmetric bracket $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ satisfying the Jacobi identity:*

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0.$$

The matrix exponential mapping from a matrix Lie algebra to a matrix Lie-group is also very important:

Definition 2.3. *Define the exponential mapping* expm: $\mathfrak{g} \to \mathcal{G}$ *by*

$$\mathrm{expm}(A) = \sum_{j=0}^{\infty} \frac{A^j}{j!}.$$

It is noted that $\mathrm{expm}(O) = I$, and that if A is sufficiently near $O \in \mathfrak{g}$, the exponential mapping has a smooth inverse given by the matrix logarithm $\mathrm{logm} : \mathcal{G} \to \mathfrak{g}$.