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PENALTY-FACTOR-FREE STABILIZED NONCONFORMING FINITE ELEMENTS FOR SOLVING STATIONARY NAVIER-STOKES EQUATIONS*

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Abstract

Two nonconforming penalty methods for the two-dimensional stationary Navier-Stokes equations are studied in this paper. These methods are based on the weakly continuous P_1 vector fields and the locally divergence-free (LDF) finite elements, which respectively penalize local divergence and are discontinuous across edges. These methods have no penalty factors and avoid solving the saddle-point problems. The existence and uniqueness of the velocity solution are proved, and the optimal error estimates of the energy norms and L^2-norms are obtained. Moreover, we propose unified pressure recovery algorithms and prove the optimal error estimates of L^2-norm for pressure. We design a unified iterative method for numerical experiments to verify the correctness of the theoretical analysis.

Mathematics subject classification: 65N30, 76D05.
Key words: Stationary Navier-Stokes equations, Nonconforming finite elements, Penalty stabilization methods, DG methods, Locally divergence-free.

1. Introduction

It is well-known that the difficulty of solving Navier-Stokes equations (or Stokes-type) with conforming mixed finite element methods is to satisfy the inf-sup stability condition and enforce the locally divergence free (LDF) property of finite element spaces. The low-order and equal-order elements do not satisfy the inf-sup condition, e.g., P_1-P_0, P_1-P_1, although P_2-P_0 satisfies, there's no optimal error estimate. The discrete velocities of the incompressible fluids have divergence-free property [20], but not accurately divergence-free property in each element.

Crouzeix and Raviart [9] proposed the CRP1 nonconforming finite element methods to solve the Stokes problems, then Temam [19] extended these methods to the stationary Navier-Stokes equations. The advantages of CRP1 nonconforming finite element are the locally divergence-free property inside each element and the weak continuity of the edge midpoint, but it's not readily available to construct global basis functions. In addition, Baker et al. [1] studied the Stokes problems by using the locally divergence-free (LDF) finite element methods, it's noteworthy that the basis functions for these methods are easy to construct but lack continuity between elements.

The locally divergence-free Crouzeix-Raviart nonconforming P_1 vector fields (namely CRP1 nonconforming finite elements) were used in the stationary Stokes equations [9] and time-harmonic Maxwell equations [3]. Two new nonconforming finite elements have been developed

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by relaxing the constraints of CRP1, that is, weakly continuous P_1 finite elements and locally divergence-free (LDF) finite elements. Firstly, it's relatively simple to define weakly continuous P_1 finite elements, which have been applied to two-dimensional curl-curl [5] and grad-div problems [2]. Secondly, approximate properties and basis functions of the locally divergence-free (LDF) finite elements have been studied in [18], and the curl-curl problems [4], Maxwell equations [8] and stationary Navier-Stokes equations [15, 16] have been solved by using this approach. Related research has confirmed the potential of the above two nonconforming finite elements. On this basis, two new discontinuous Galerkin methods, namely the LDP method and the DG-LDF method, are proposed to solve the two-dimensional Stokes problems without penalty factors [17].

In this paper, the two approaches in [17] are generalized to the nonlinear incompressible stationary Navier-Stokes equations. We prove the existence and uniqueness of the velocity solution, and obtain the optimal error estimates. Due to different definitions of unified form, the forms of error estimates are also different from [17]. However, it can be seen from the numerical experiments that the optimal convergence orders of the energy norms and L^2-norms for the velocities are the same as [17]. First of all, we use the method of [20], which only focuses on solving the velocity of the fluid, the strengths of this method are to avoid the saddle-point problem and thus the inf-sup condition. Then, the method in [11, 21] is applied to establish the pressure recovery algorithm according to the calculated velocity. Based on the two velocity schemes, unified pressure recovery algorithms are proposed and the optimal error estimate of L^2-norm for the pressure is analyzed. A unified iterative method is designed to verify the correctness and efficiency of the theoretical analysis.

This paper is organized as follows. In section 2, we introduce basic notations for Sobolev spaces and two weak formulations for stationary Navier-Stokes equations. In section 3, we propose the LDP and DG-LDF schemes and prove the existence and uniqueness of the velocity solution in section 4. In section 5, optimal convergence orders for the energy norms and L^2-norms for the velocity are obtained. In section 6, unified pressure recovery algorithms are proposed and the optimal error estimate of L^2-norm for the pressure is proved. In section 7, we propose a unified iterative method and carry out numerical experiments, and we give some concluding remarks in section 8.

2. Notations and Weak Formulations

We consider the following stationary Navier-Stokes equations:

$$\begin{cases} -\nu\Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^d (d = 2, 3)$ is a bounded domain, $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}) \in \mathbb{R}^d$ is the fluid velocity, $p = p(\boldsymbol{x}) \in \mathbb{R}$ is the pressure, $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x}) \in \mathbb{R}^d$ is an external body force, $\nu > 0$ is the viscosity coefficient.

Throughout the paper, for real-valued functions, $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$ denote the k-order Sobolev spaces, and these spaces are endowed with norms $||\cdot||_{W^{k,p}(\Omega)} = ||\cdot||_{k,p,\Omega}$ and semi-norms $|\cdot|_{W^{k,p}(\Omega)} = |\cdot|_{k,p,\Omega}$. When $k = 1, p = 2$, $W^{1,2}(\Omega) = H^1(\Omega)$ and $W_0^{1,2}(\Omega) = H_0^1(\Omega)$. When $k = 0, p = 2$, $W^{0,2}(\Omega) = L^2(\Omega)$ is equipped with the L^2 inner product $(\cdot,\cdot)$ and L^2-norm $\|\cdot\|_0$. Similarly, for vector-valued functions: $H^1(\Omega)^d = \boldsymbol{H}^1(\Omega), H_0^1(\Omega)^d = \boldsymbol{H}_0^1(\Omega), L^2(\Omega)^d = \boldsymbol{L}^2(\Omega)$. Let $M = L_0^2(\Omega) = \{q \in L^2(\Omega) : (q, 1) = 0\}, H(\text{div}0) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) : \nabla\cdot\boldsymbol{v} = 0\}$, $\boldsymbol{X} = H_0^1(\Omega)^d$,