Journal of Computational Mathematics
Vol.40, No.5, 2022, 781–797.

http://www.global-sci.org/jcm
doi:10.4208/jcm.2101-m2020-0075

ANALYSIS OF A FULLY DISCRETE FINITE ELEMENT METHOD FOR PARABOLIC INTERFACE PROBLEMS WITH NONSMOOTH INITIAL DATA*

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Abstract

This article concerns numerical approximation of a parabolic interface problem with general L^2 initial value. The problem is discretized by a finite element method with a quasi-uniform triangulation of the domain fitting the interface, with piecewise linear approximation to the interface. The semi-discrete finite element problem is furthermore discretized in time by the k-step backward difference formula with $k = 1, \ldots, 6$. To maintain high-order convergence in time for possibly nonsmooth L^2 initial value, we modify the standard backward difference formula at the first $k-1$ time levels by using a method recently developed for fractional evolution equations. An error bound of $\mathcal{O}(t_n^{-k}\tau^k + t_n^{-1}h^2|\log h|)$ is established for the fully discrete finite element method for general L^2 initial data.

Mathematics subject classification: 65M60, 65N30, 65N15.
Key words: Parabolic interface problem, Finite element method, Backward difference formulae, Error estimate, Nonsmooth initial data.

1. Introduction

Elliptic and parabolic problems with discontinuous coefficients often appear in material sciences and fluid dynamics. Examples include conductivity of steel in heat diffusion, underground water flow, and fluid flow in porous media [2,12,19]. When the interface is sufficiently smooth, the solution of the interface problem is also smooth enough in each separate domains, but it generally has lower global regularity because of the non-homogeneous jumps across the interface. Various numerical methods for solving such kind of problems have been well studied over the past several decades.

Babuska [1] first studied elliptic interface problems with a smooth interface using classical finite element method (FEM). By reformulating the problem as a minimization problem with a quadratic functional, he obtained optimal-order error estimate in H^1 norm. Chen and Zou [3] considered this problem on a polygonal domain with C^2 interface using classical FEM and established almost optimal-order error estimate (optimal up to a logarithmic factor) in both H^1 and L^2 norms. Huang and Zou [6] proposed a mortar element method for elliptic interface

* Received March 26, 2020 / Revised version received December 1, 2020 / Accepted January 27, 2021 / Published online May 30, 2021 /
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problems and proved optimal-order error estimates under some mortar conditions. Sinha and Deka in [16–18] studied FEMs for elliptic interface problem and established an optimal-order error estimate in the H^1 norm, together with an almost optimal-order error estimate in the L^2 norm. Li et al. [11] extended the analysis in [3] to higher-order FEMs and proved optimal-order error estimates in both H^1 and L^2 norms.

Analysis of parabolic interface problems has also been considered by many authors. In particular, Chen and Zou [3] considered backward Euler method in time and piecewise linear FEM in space. They proved the following error estimates:

$$\|u - u_{\tau,h}\|_{L^2(0,T;L^2(\Omega))} \le C\left(\tau + h^2|\log h|\right),$$
$$\|u - u_{\tau,h}\|_{L^2(0,T;H^1(\Omega))} \le C\big(\tau + h|\log h|^{\frac{1}{2}}\big),$$

under the assumption that the initial value is in $H_0^1(\Omega)$, where $u_{\tau,h}$ is the piecewise constant interpolation of the numerical solutions in time, i.e.,

$$u_{\tau,h}(x,t) = u_h^n(x), \quad \forall t \in \left(t^{n-1}, t^n\right], \quad n = 1, \dots, N.$$

Sinha and Deka in [17] proved an error bound $\mathcal{O}(\tau+h)$ in the $L^2(0,T;H^1(\Omega))$ norm for a discontinuous Galerkin time-stepping method under the assumption that the initial value is in $H_0^1(\Omega)$. The same convergence order was proved for a backward Euler time-stepping method in [18]. In [4], Deka and Ahmed obtained optimal-order error estimate $\mathcal{O}(h^2)$ in the $L^2(0,T;L^2(\Omega))$ norm for the semi-discrete FEM with initial value $u_0 \in H_0^1(\Omega)$. More recently, Deka and Roy [5] proved an optimal-order error estimate at a fixed time level, i.e.,

$$\begin{aligned}&\|u(t_n) - u_h^n\|_{L^2(\Omega)} \\ &\le C(\tau + h^{r+1})\Big(\|u(0)\|_{H^{r+1}(\Omega)} + \|\partial_t u\|_{L^2(0,T;H^{r+1}(\Omega))} + \|\partial_{tt} u\|_{L^2(0,T;L^2(\Omega))}\Big)\end{aligned} \tag{1.1}$$

for a backward Euler method with weak Galerkin FEM in space with polynomial degree $r \ge 1$. In the case of piecewise linear FEM (i.e., $r = 1$), the error estimate requires the initial value to be in $H^2(\Omega)$. As far as we know, no error analysis has been done for parabolic interface problems when the initial value is only in $L^2(\Omega)$. In this case, the solution does not have the regularity required on the right-hand side of (1.1). Hence, we cannot expect the numerical solutions to have optimal-order convergence uniformly in time or in the $L^2(0,T;L^2(\Omega))$ norm.

Recently, many efforts have been made to develop accurate and efficient methods for time-fractional evolution equations with nonsmooth initial data using the Laplace transform technique [7–9, 20, 21] and the convolution quadrature developed by Lubich [13]. These techniques allow people to develop high-order convergent methods with a uniform time step size for problems whose solutions may possess singularity at time $t = 0$. More recently, this technique was extended to the construction of high-order symmetrized and decoupled methods for parabolic systems with nonsmooth data [10] and tempered fractional Feynman–Kac equation [14].

The objective of this paper is to develop a fully discrete FEM for parabolic interface problems with possibly nonsmooth initial value $u_0 \in L^2(\Omega)$, and present rigorous analysis for convergence of numerical solutions by utilizing the Laplace transform technique that is widely used in analysis of time-fractional equations. We prove that the proposed k-step method, for a parabolic interface problem with an initial value $u_0 \in L^2(\Omega)$, can have almost second-order convergence in space and k-th-order convergence in time, with an error bound

$$\|u(t_n) - u_h^n\|_{L^2(\Omega)} \le C\Big(t_n^{-k}\tau^k + t_n^{-1}h^2|\log h|\Big).$$