Journal of Computational Mathematics
Vol.40, No.5, 2022, 818–838.
http://www.global-sci.org/jcm
doi:10.4208/jcm.2102-m2020-0211

ANALYSIS OF A MULTI-TERM VARIABLE-ORDER TIME-FRACTIONAL DIFFUSION EQUATION AND ITS GALERKIN FINITE ELEMENT APPROXIMATION*

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Abstract

In this paper, we study the well-posedness and solution regularity of a multi-term variable-order time-fractional diffusion equation, and then develop an optimal Galerkin finite element scheme without any regularity assumption on its true solution. We show that the solution regularity of the considered problem can be affected by the maximum value of variable-order at initial time $t = 0$. More precisely, we prove that the solution to the multi-term variable-order time-fractional diffusion equation belongs to $C^2([0, T])$ in time provided that the maximum value has an integer limit near the initial time and the data has sufficient smoothness, otherwise the solution exhibits the same singular behavior like its constant-order counterpart. Based on these regularity results, we prove optimal-order convergence rate of the Galerkin finite element scheme. Furthermore, we develop an efficient parallel-in-time algorithm to reduce the computational costs of the evaluation of multi-term variable-order fractional derivatives. Numerical experiments are put forward to verify the theoretical findings and to demonstrate the efficiency of the proposed scheme.

Mathematics subject classification: 35B65, 35R11, 65M12, 65M60.
Key words: Variable-order, Multi-term time-fractional diffusion equation, Solution regularity, Galerkin finite element, Parareal method.

1. Introduction

In recent years, it was shown that fractional partial differential equations (FPDEs) possess more powerful modeling capacity and offer better descriptions than their integer-order analogues for many important physical applications [2, 13, 19, 23, 29]. We can also explain this by the relation between the mean square displacement (MSE) of particles and time. Instead of the linear dependence of integer-order PDEs, the MSE corresponding to FPDEs is nonlinearly dependent on time which has been verified by many mathematical derivation and experimental observations [3, 20, 28]. However, FPDEs also exhibit some new issues that do not appear in the treatment of integer-order PDEs, such as the nonphysical singularity of solution and nonlocal property of fractional derivatives [5, 9, 12, 25].

* Received August 14, 2020 / Revised version received January 13, 2021 / Accepted February 2, 2021 / Published online August 16, 2021 /
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Up to now, extensive works have been devoted to mathematical/numerical analysis of the constant-order FPDEs [8,10,14,30,34,38]. However, there are also many complicated physical phenomena that can not be properly characterized by the constant-order FPDEs, e.g., the ultra-slow diffusion, accelerating superdiffusion or other complex anomalous diffusion processes. Numerical experiments show that variable-order FPDEs can provide a better fitting to the experimental data, if the diffusion behavior of solute particles transported in heterogeneous medium changes over time, space or other independent quantities [26,27,29]. As a consequence, the variable-order FPDEs provide a more comprehensive modeling approach and show a way to link the anomalous diffusion to normal diffusion [27].

Although various numerical methods have been developed for solving variable-order F-PDEs [39,43], however, the optimal error estimate in most of the existing works were derived by assuming that the true solutions have sufficient smoothness. It was gradually realized that the solution regularity of variable-order FPDEs is more complicated than constant-order F-PDEs [40,42]. Recently, there were some works on the mathematical analysis of variable-order FPDEs. The existence and uniqueness were established for a variable-order FPDE, in which the variable-order was assumed to be a piecewise constant function of time [32]. In [33], Wang and Zheng proved the well-posedness and regularity for a variable-order linear time-fractional diffusion equation (tFDE) and showed that the solution regularity depends on the behavior of the variable-order at initial time. Subsequently, they further derived the regularity result for a variable-order space-fractional diffusion equation and developed an optimal indirect method discretized on uniform or graded meshes [41].

The multi-term FPDEs have attracted a lot of attention due to their improved modeling capacity than single-term FPDEs [11,18]. Motivated by the work for single-term variable-order tFDE in [33,40], we investigate the well-posedness and regularity of the following multi-term variable-order tFDE [16]

$$\partial_t u + \sum_{m=1}^{M} k_m(t)\,\partial_t^{\alpha_m(t)} u + \mathcal{A}u = f(\boldsymbol{x},t), \quad (\boldsymbol{x},t) \in \Omega\times(0,T]; \tag{1.1a}$$

$$u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), \qquad \boldsymbol{x}\in\Omega; \tag{1.1b}$$

$$u(\boldsymbol{x},t) = 0, \qquad (\boldsymbol{x},t)\in\partial\Omega\times[0,T]. \tag{1.1c}$$

Here $\Omega \subset \mathbb{R}^d$ $(d = 1,2,3)$ is a simply-connected bounded domain with piecewise smooth boundary $\partial\Omega$ and convex corners, $\boldsymbol{x} := (x_1,\cdots,x_d)$, the operator $\mathcal{A} := -\nabla\cdot\big(\boldsymbol{A}(\boldsymbol{x})\nabla\big)$ with $\nabla := \big(\partial/\partial x_1,\cdots,\partial/\partial x_d\big)^T$ and $\boldsymbol{A}(\boldsymbol{x}) := (a_{i,j}(\boldsymbol{x}))_{i,j=1}^d$. The variable-order time-fractional differential operator $\partial_t^{\alpha_m(t)}$ is defined by [17,26,27]

$$\partial_t^{\alpha_m(t)} g := {}_0I_t^{1-\alpha_m(t)}\partial_t g, \qquad {}_0I_t^{1-\alpha_m(t)} g := \frac{1}{\Gamma(1-\alpha_m(t))}\int_0^t \frac{g(s)ds}{(t-s)^{\alpha_m(t)}}. \tag{1.2}$$

In this paper, we focus on the well-posedness and regularity of the multi-term variable-order tFDE (1.1). We prove that the second-order derivative of the solution with respect to time is continuous, provided that the maximum value has integer limit and the data has sufficient smoothness. We then develop a Galerkin finite element method for solving the considered problem, numerical analysis shows that the proposed Galerkin method has first-order accuracy in time and second-order accuracy in space without any regularity assumptions on its true solution. To improve the computational efficiency, we develop a parallel-in-time Galerkin algorithm