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STABILIZED NONCONFORMING MIXED FINITE ELEMENT METHOD FOR LINEAR ELASTICITY ON RECTANGULAR OR CUBIC MESHES*

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Abstract

Based on the primal mixed variational formulation, a stabilized nonconforming mixed finite element method is proposed for the linear elasticity on rectangular and cubic meshes. Two kinds of penalty terms are introduced in the stabilized mixed formulation, which are the jump penalty term for the displacement and the divergence penalty term for the stress. We use the classical nonconforming rectangular and cubic elements for the displacement and the discontinuous piecewise polynomial space for the stress, where the discrete space for stress are carefully chosen to guarantee the well-posedness of discrete formulation. The stabilized mixed method is locking-free. The optimal convergence order is derived in the L^2-norm for stress and in the broken H^1-norm and L^2-norm for displacement. A numerical test is carried out to verify the optimal convergence of the stabilized method.

Mathematics subject classification: 65N15, 65N30.
Key words: Mixed finite element method, Nonconforming rectangular or cubic elements, Elasticity, Locking-free, Stabilization.

1. Introduction

For the linear elasticity problem, the pure displacement formulation is a very common one. However, the so-called locking phenomenon may appear when this formulation is numerically solved in the nearly incompressible or incompressible case. Therefore, some locking-free finite element methods (FEMs) for this pure displacement formulation have been developed, see e.g. [7,12,19,25,28,40]. For example, Brenner and Sung developed a locking-free nonconforming FEM in [7] by using the well-known Crouzeix-Raviart (CR) element [15]. Therein, the elasticity operator was split into the gradient part and the divergence part with appropriate coefficients. However, the CR element is not stable for the elasticity operator, since it does not fulfill the discrete Korn's inequality. To overcome this problem, Hansbo and Larson proposed a stabilized method for the CR element in [19]. In other words, they added a jump penalty term for the displacement to get a locking-free stabilized nonconforming FEM. The optimal convergence was

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proved in a mesh-dependent norm. Similar stabilization technique was also done in [18] for the nonconforming quadrilateral element [33].

By contrast, the mixed formulation, where both the stress and displacement are simultaneously solved, is preferable to the pure displacement one, since the stress is usually a physical quantity of primary interest. For the Hellinger-Reissner mixed formulation, there exist two possibilities to apply the derivatives to the displacement or stress, that lead to the primal mixed formulation and the dual mixed formulation. For the second one, it requires that the stress tensor element is symmetric with continuous normal components and satisfies the discrete inf-sup condition. it is very difficult to construct such elements, so most of these elements are quite complicated, especially in three dimensions, see e.g. [1, 3–5, 24]. We also mention the further development of stable conforming elements from the references [20–23], where a new class of stable conforming elements called the Hu-Zhang element is proposed. In order to use common elements, many stabilized methods are proposed for the dual mixed method, see [9–11, 27, 35, 38] and therein the references. For more discussions on other methods, such as nonconforming mixed FEMs, mixed FEMs with weaker symmetry and discontinuous Galerkin methods, we refer the readers to the related references mentioned above.

On the other hand, the primal mixed formulation is easy to discretize, because it does not need the continuity of stress and the discrete inf-sup condition can be easily satisfied in this case. However, this primal mixed formulation usually leads to the standard FEMs suffering locking as the pure displacement formulation, unless special techniques are applied. Based on the primal mixed formulation, Franca and Hughes proposed two classes of stabilized conforming mixed FEMs for elasticity, called circumventing Babuška-Brezzi condition method (CBB) and satisfying Babuška-Brezzi condition method (SBB), see [16]. For the CBB method, the discontinuous or continuous piecewise polynomial space can be used for the stress approximation and the continuous piecewise polynomial space for the displacement approximation. For the SBB method, only the discontinuous piecewise polynomial space is used for the stress approximation. The CBB method is convergent, provided the method is employed in the compressible case. The SBB method is uniformly convergent for the nearly incompressible or incompressible case. Recently, a stabilized nonconforming mixed FEM was shown to be locking-free in [37] where the displacement is approximated by the CR element and the stress by piecewise constants. Therein, the jump penalty term is added for the displacement to get the stability of the formulation. The uniform convergence was proved based on the fact that the discrete space for the CR element contains the subspace of continuous piecewise linear functions. We mention that for finite Lamé constant λ, the stabilized nonconforming mixed method in [37] is reduced to the one in [19] for the pure displacement formulation.

We mention that the assumed stress hybrid FEM on quadrilateral or hexahedral meshes is closely related to the stabilized methods proposed in [16, 37]. The pioneering work of assumed stress hybrid FEM was presented in [30], where the assumed stress field is assumed to satisfy the homogeneous equilibrium equations pointwisely. Then a hybrid stress quadrilateral element was constructed in [32], where the isoparametric bilinear element is used for the displacement approximation and a discontinuous piecewise polynomial space for the stress approximation. By eliminating the stress parameters at the element level, the hybrid stress method [32] finally leads to a symmetric and positive definite discrete system involving only displacement unknowns. The uniform convergence analysis for the hybrid stress method can be found in [26, 36]. For more works on hybrid FEMs for elasticity, please see [31, 39] and the references therein.

In this paper, we propose a stabilized nonconforming mixed FEM to discretize the primal