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AN L^∞ SECOND ORDER CARTESIAN METHOD FOR 3D ANISOTROPIC INTERFACE PROBLEMS*

Baiying Dong
School of Mathematics and Statistics, NingXia University, Yinchuan 750021, China
School of Mathematics and Computer Science, NingXia Normal University, Guyuan 756000, China
Email: baiying_dong@nxnu.edu.cn
Xiufeng Feng[1)]
School of Mathematics and Statistics, NingXia University, Yinchuan 750021, China
Email: xf_feng@nxu.edu.cn
Zhilin Li
Department of Mathematics, North Carolina State University, Raleigh, NC 27695-8205, USA
Email: zhilin@ncsu.edu

Abstract

A second order accurate method in the infinity norm is proposed for general three dimensional anisotropic elliptic interface problems in which the solution and its derivatives, the coefficients, and source terms all can have finite jumps across one or several arbitrary smooth interfaces. The method is based on the 2D finite element-finite difference (FE-FD) method but with substantial differences in method derivation, implementation, and convergence analysis. One of challenges is to derive 3D interface relations since there is no invariance anymore under coordinate system transforms for the partial differential equations and the jump conditions. A finite element discretization whose coefficient matrix is a symmetric semi-positive definite is used away from the interface; and the maximum preserving finite difference discretization whose coefficient matrix part is an M-matrix is constructed at irregular elements where the interface cuts through. We aim to get a sharp interface method that can have second order accuracy in the point-wise norm. We show the convergence analysis by splitting errors into several parts. Nontrivial numerical examples are presented to confirm the convergence analysis.

Mathematics subject classification: 65M06, 76M20, 65N06.
Key words: 3D anisotropic PDE, Cartesian meshes, Finite element method, Finite difference method, Maximum preserving IIM, Convergence analysis.

1. Introduction

In this paper, we develop a finite element-finite difference method for three dimensional (3D) anisotropic elliptic partial differential equations (PDEs) involving finite number of non-overlapping interfaces across which the coefficients of the PDE may be discontinuous, and the source term may be discontinuous and even can have a singular source term corresponding to a jump in the flux or the solution. The problem is described as follows,

$$
\begin{aligned}
& -\nabla\cdot\Big(\mathbf{A}(\mathbf{x})\nabla u(\mathbf{x})\Big)+\sigma(\mathbf{x})u(\mathbf{x})=f(\mathbf{x}), \quad \mathbf{x}=(x,y,z)\in\Omega\setminus\Gamma,\ \Omega=\Omega^+\cup\Omega^-, \\
& u(\mathbf{x})=u_0(\mathbf{x}), \quad \mathbf{x}=(x,y,z)\in\partial\Omega,
\end{aligned}
\tag{1.1}
$$

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1) Corresponding author

where the coefficient matrix $\mathbf{A}(\mathbf{x}) \in C^1(\Omega \setminus \Gamma)$ is a 3×3 symmetric positive definite (SPD) matrix,

$$\mathbf{A}(\mathbf{x}) = \begin{pmatrix} A_{11}(\mathbf{x}) & A_{12}(\mathbf{x}) & A_{13}(\mathbf{x}) \\ A_{12}(\mathbf{x}) & A_{22}(\mathbf{x}) & A_{23}(\mathbf{x}) \\ A_{13}(\mathbf{x}) & A_{23}(\mathbf{x}) & A_{33}(\mathbf{x}) \end{pmatrix}, \qquad \mathbf{x} \in \Omega. \tag{1.2}$$

We define the $\mathbf{A}^+$ and $\mathbf{A}^-$ as the restrictions of $\mathbf{A}$ on Ω^+ and Ω^-, respectively,

$$\mathbf{A} = \begin{cases} \mathbf{A}^+(\mathbf{x}), & \text{if} \quad \mathbf{x} \in \Omega^+, \\ \mathbf{A}^-(\mathbf{x}), & \text{if} \quad \mathbf{x} \in \Omega^-, \end{cases} \tag{1.3}$$

where $\mathbf{A}^\pm(\mathbf{x}) \in C^1(\Omega^\pm)$. Since we use both finite element and finite difference discretization, we assume that $\sigma(\mathbf{x}) \geq 0$; $f(\mathbf{x}) \in C(\Omega \setminus \Gamma)$, and the interface Γ is C^2 within the domain Ω, see Fig. 1.1 for an illustration. We allow both of the solution and the flux to be discontinuous,

$$[u](\mathbf{X}) = w(\mathbf{X}), \quad [\mathbf{A}\nabla u \cdot \mathbf{n}](\mathbf{X}) = Q(\mathbf{X}), \quad \mathbf{X} = (X, Y, Z) \in \Gamma, \tag{1.4}$$

where $\mathbf{n}(\mathbf{X})$ is the unit normal direction at a point $\mathbf{X}$ on the interface pointing to the Ω^+ side. For the regularity requirement, we assume that $w \in C^2(\Gamma)$, and $Q \in C^1(\Gamma)$. The above two jump conditions along the boundary condition make the problem well-posed. The jumps on the interface, such as $[u](\mathbf{X})$ and $[\mathbf{A}\nabla u \cdot \mathbf{n}](\mathbf{X})$, are defined as the differences of the limiting values from different sides of the interface; for example,

$$[u](\mathbf{X}) = \lim_{\mathbf{x}\to\mathbf{X}, \mathbf{x}\in\mathbf{\Omega}^+} u(\mathbf{x}) - \lim_{\mathbf{x}\to\mathbf{X}, \mathbf{x}\in\mathbf{\Omega}^-} u(\mathbf{x}) = u^+ - u^-.$$

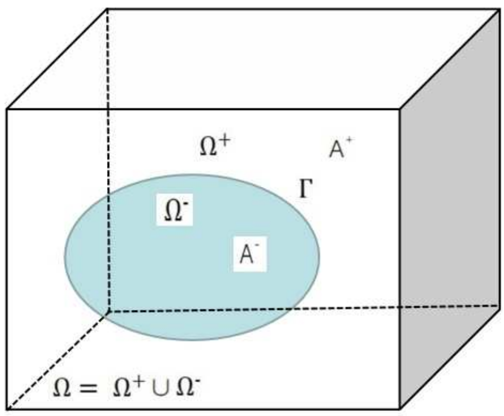

Fig. 1.1. A diagram of the anisotropic interface problem: a rectangular domain with a closed smooth interface (surface).

The existence and uniqueness of the solution is well-known based on the Lax-Milgram lemma, see for example [1]. The Galerkin finite element method can be applied to solve the problem numerically if a body-fitted 3D mesh (unstructured) can be generated, which is non-trivial and maybe time consuming. However, sometimes Cartesian methods are preferred for number of reasons. In a Cartesian mesh method, we do not need to generate the mesh. The resulting linear system of equations can be solved by structured fast solvers; A Cartesian mesh method is often easier to be combined with other existing packages. Challenges with Cartesian methods include how to get accurate discretization near or on the interface and carry out the convergence analysis. There are limited Cartesian methods for anisotropic interface problems, mostly based on Galerkin finite element method, for example, the immersed finite